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Speculative Price Dynamics in a Heterogeneous Agent Model

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This paper proposes an agent model of financial markets and analyzes factors leading to speculative bubbles and speculative chaos of the asset price. A financial market is thought to contain two typical types of traders: fundamentalists and chartists who try to maximize their utility. It is shown that the nonlinearity of the excess demand functions, which are derived as a result of the traders' utility maximization, might generate speculative bubbles and speculative chaos of the asset price.

KEY WORDS: speculative bubble; chaos; market; heterogeneous agent; utility.

INTRODUCTION

A distinguishing characteristic of economic and financial systems is that the system dynamics are affected by the heterogeneity of agents who constitute the system. It follows that differences in trading strategies of traders in financial markets as well as differences in the trader's expectations about the situation of the system in the future may generate different dynamic paths. There is already a growing literature attempting to model heterogeneity in traders' trading strategies which may lead to market instability and complicated dynamics, such as cycles or even chaotic fluctuations in financial markets (Brock & Hommes, 1997, 1998; Chen & Yeh, 2001; Chiarella, 1992; Chiarella, Dieci, & Gardini, in press; Day & Hwang,

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1990; DeGrauwe, DeWachter, & Embrechts, 1993; Lux, 1995, 1998; Kaizoji, 2001).[3]

Our previous work (Kaizoji, 2001) proposed a heterogeneous agent model of a stock market which lead to complicated endogenous fluctuations. We concentrated on showing that the distributions of relative price changes generated from the model have the fat tails which is a stylized fact observed in almost all the financial markets, and paid scant attention by the analysis of the dynamics of markets. In the present paper, we investigate the dynamic behavior in more detail. To this end we simplify the model presented in Kaizoji (2001) in a respect. While the previous model included a learning process on choice of strategies, learning by traders will be omitted in the present model. The model is reduced to a two-dimensional nonlinear map that is mathematically tractable thanks to this simplification. From the standpoint of economic theories, it is true that learning is an important factor in order to understand a trader's behavior, but I would like to emphasize that this simplification will make possible mathematical proofs of the dynamical properties of our model including the existence of chaotic attractors.

We think of a simple financial market as consisting of a market maker which mediates the risky asset, and two typical types of traders: *fundamentalists* who are forming accurate expectations on the fundamental value of the asset, and *chartists* who base their trading decisions on an analysis of past price trends. We shall demonstrate the regions in the parameter space in which the fundamental price that is the unique equilibrium point of the map is locally stable, and indicate the bifurcations that the equilibrium point undergoes when the key parameters, such as the strength of nonlinearity of excess demand functions, change continuously. We in particular show the following:

1. When the nonlinearity of the fundamentalist excess demand function is sufficiently strong, the stationary chaos will be observed, that is, irregular fluctuations of the asset price around the fundamental value occur.
2. When the nonlinearity of the chartist excess demand function is sufficiently strong, the *non-stationary chaos* or *speculative bubbles* will be observed, that is, irregular fluctuations of the asset price around the trend.

The next section of the paper presents the heterogeneous agent model briefly. The third section investigates the dynamical properties of the price dynamics. The final section sums up briefly.

[3]Early studies of heterogeneous agent models are Zeeman (1974), and Frankel and Froot (1988).

THE MODEL

Let us consider the market of a risky asset that is composed of two groups of traders having the different trading strategies; *fundamentalists* and *chartists*.

Fundamentalists

Fundamentalists are assumed to have a reasonable knowledge of the fundamental value of the risky asset. Fundamentalist excess demand for the risky asset is given by:

$$x_t^f = \exp(\alpha(p^* - p_t)) - 1, \quad \alpha > 0, \tag{1}$$

where p^* is the fundamental value, p_t the price of the risky asset at time t, and α the parameter that denotes the strength of the nonlinearity of the fundamentalist excess demand function (1). For fundamentalists it is assumed that if the price p_t is below the fundamental value p^*, then they try to buy the risky asset, because they think that the risky asset is undervalued, and if the price is above the fundamental value, then they try to sell the risky asset, because they believe that the risky asset is overvalued.

The fundamentalist excess demand function (1) is derived in a one-period utility optimizing framework. The technical details of the derivation of (1) within a utility maximaizing framework are given in Kaizoji (2001).

Chartists

The difference between chartists and fundamentalists is that chartist estimate a trend to the price change while fundamentalists calculate the fundamental value. Thus, the chartists' decision making is based upon their expected price. Similarly to the example of fundamentalists chartist excess demand function is given by:

$$x_t^c = \exp\left(\beta\left(p_{t+1}^e - p_t\right)\right) - 1, \quad \beta > 0, \tag{2}$$

The chartist excess demand function (2) means that chartists try to buy the risky asset when they anticipate that the price will rise at the next period, and inversely that they try to sell the risky asset when they expect the price will fall at the next period.

Chartists are assumed to form their expectation of the price of the risky asset according to the simple adaptive scheme:

$$p_{t+1}^e = p_t^e + \mu\left(p_t - p_t^e\right), \tag{3}$$

where the parameter $\mu (0 < \mu \geq 1)$ is an *error correction coefficient.*

The Adjustment Process of the Price

Let us now consider the adjustment process of the price in the market. We assume that the existence of a market-maker who mediates the trading. If the excess demand in period t is positive (negative), the market maker raises (reduces) the price for the next period $t+1$. This process of adjustment of the price can be given as

$$p_{t+1} - p_t = \theta n\big[(1-\kappa)x_t^f + \kappa x_t^c\big], \tag{4}$$

where θ denotes the speed of the adjustment of the price decided by the market maker, and n the total number of traders and κ, the proportion of chartists. We simply assume that the proportion κ is constant over time.

DYNAMICS

The aim of this section is to describe some of the possible types of dynamical behavior of the model described above when the two parameters α and β, that is denote the strength of the nonlinearity of the excess demand functions, are allowed to vary. Substituting (3) to (4), the model can be rewritten as a fist order difference equation system

$$\begin{aligned} p_{t+1} &= p_t + \theta n\big[(1-\kappa)(\exp(\alpha(p^* - p_t)) - 1) + \kappa\big(\exp\big(\beta(1-\mu) \\ &\qquad \times \big(p_t^e - p_t\big)\big) - 1\big)\big], \\ p_{t+1}^e &= (1-\mu)p_t^e + \mu p_t. \end{aligned} \tag{5}$$

It is clear that the two dimensional map (5) has the unique equilibrium with $p_t^e = p_t = p^*$, namely $(\bar{p}^e, \bar{p}) = (p^*, p^*)$.

The Local Stability Conditions

The local stability analysis of the equilibrium $(\bar{p}^*, \bar{p})$ is performed via the evaluation of the two eigenvalues of the Jacobian matrix of the map:

$$J = \begin{bmatrix} 1 - \theta n((1-\kappa)\alpha + \kappa\beta(1-\mu)) & \kappa\beta\theta n(1-\mu) \\ \mu & (1-\mu) \end{bmatrix} \tag{6}$$

at (p^*, p^*), and has $D = (1-\mu)[1 - \theta n(\alpha(1-\kappa) + \beta\kappa)]$ and $T = (1-\mu) \times [1 - \theta n(\alpha(1-\kappa) + \beta(1-\mu)\kappa)]$ where D and T dente the trace and determinant of the Jacobian matrix. The eigenvalues are obtained by solving the following characteristic equation: $c(\lambda) = \lambda^2 - T\lambda + D = 0$. The eigenvalues are $\lambda = T/2 + \pm\sqrt{\Delta}/2$ where $\Delta \equiv T^2 - 4D$. As it is well known a sufficient condition for the local stability consists in the following system of inequalities: (i) $c(1) = 1 - T + D > 0$, (ii) $1 + T + D > 0$, and (iii) $1 - T + D > 0$, giving necessary and sufficient conditions for the two eigenvalues to be inside the unit circle of the complex plane. Elementary computations show that for our map (5) the condition is given as:

$$\alpha < \frac{2}{\theta n(1-\kappa)} - \frac{2(1-\mu)\kappa}{(2-\mu)(1-\kappa)}\beta. \tag{7}$$

Figure 1 represents the region of local stability of the equilibrium in the parameter plane (α, β), $\alpha, \beta > 0$. From Eq. 7 it is follows that, starting from parameters (α, β) inside the stability region, a loss of stability may occur via a flip bifurcation, when crossing the curve

$$\alpha = \frac{2}{\theta n(1-\kappa)} - \frac{2(1-\mu)\kappa}{(2-\mu)(1-\kappa)}\beta. \tag{8}$$

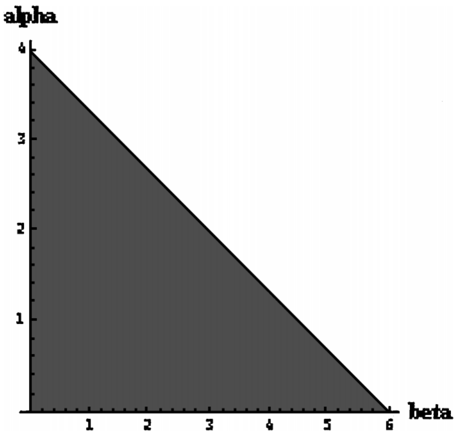

Fig. 1. The shaded region shows the domain of stability of the equilibrium point (p^*, p^*) in the plane of the parameters α (strength of non-linearity of the chartist excess demand function) and β (strength of non-linearity of the chartist excess demand function). Figure 1 is obtained for $\theta = 0.001$, $n = 1000$, $\kappa = 0.5$, and $\mu = 0.5$. The stability can be lost only via a Flip bifurcation as either α or β is increased.

As we can see from Fig. 1, the equilibrium becomes unstable as either the parameter α or β is increased. In the following subsections we highlight the impact of both α and β on the dynamics.

Stationary Chaos

As already remarked in the preceding subsection, at a given level of β the equilibrium is stable for sufficiently low values of α, but the stability is lost only via a flip bifurcation as α is increased. An increase in α, causing the flip bifurcation of the equilibrium, shall give cause to a sable 2-cycle. As α is further increased, the usual flip-cascade shall occur, and leads to chaotic dynamics, as shown in Fig. 2a. It is well known that one of the most common routes to chaos is associated with the so-called period doubling sequence. The chaotic attractor is geometrically characterized by the simultaneous presence of stretching and folding, implying that two initially close points will be projected to different locations in phase space. The presence and interaction of stretching and folding in the dynamical system can described via the *Lyapunov exponents*. The meaning of the Lyapunov exponents can be interpreted as follows: when the largest Lyapunov exponent is negative on an attractor, the attractor is stable periodic points. When the largest Lyapunov exponent is positive, the dynamical system is chaotic. Figure 2b shows calculated values of the Lyapunov exponents for the our dynamical system versus the parameter α. As can be seen from the figure, the exponents are negative for values of α lower than the critical value α. In the chaotic regime, the exponents are typically positive, but there are α with negative Lyapunov exponents, indicating the presence of stable period points.

From the mathematical point of view, theorems that provide conditions for chaos of multidimensional dynamical system were proven by Smale (1967) and Marotto (1978). Smale has proven the following.

Theorem 1 (Smale, 1967). *If F is a diffeomorphism and has a transcersal homoclinic orbit, then there exists a Cantor set $\Lambda \subset R^n$ in which F^M is topologically equivalent to the shift automorphism for some M.*

The existence of such a shift automorphism implies that within the set Λ there exists a dense collection of periodic solutions of different periods and an uncountably infinite collection of points which are asymptotically aperiodic. Thus a transversal homoclinic orbit implies a form of chaotic behavior similar to that defined by Li-Yorke (1975). Another theorem was provided by Marotto (1978):

Theorem 2 (Marotto, 1978). *If F is differentiable and has a snap-back repeller, then the dynamical system is chaotic.*

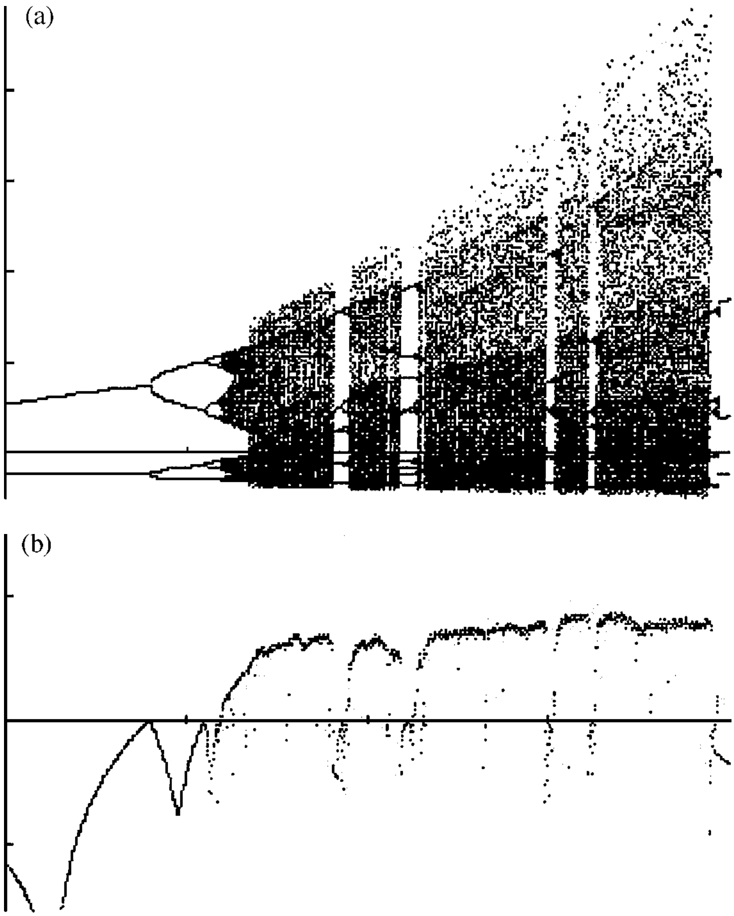

Fig. 2. (a) The bifurcation diagram is a multiple valued plot of the attracting set of the price p_t where the values of the parameter α varies smoothly from 4 to 8 with $\beta = 1, \theta = 0.001, n = 1000, \kappa = 0.5$, and $\mu = 0.5$. (b) Numerical computations of largest Lyapunov exponents, which measure the sensitive dependence on initial conditions, are plotted against increasing values of the parameter α for $\beta = 2$, $\theta = 0.001$ and $n = 1000, \kappa = 0.5$, and $\mu = 0.5$. Note that they are negative in the periodic region-indicating no sensitive dependence, and positive in the chaotic region. The sign of the largest Lyapunov exponents correlates completely with the price dynamics as shown in the bifurcation diagram.

Marotto's theorem is roughly a generalization of Li-Yorke theorem. Marotto's theorem is also closely related to Smale's conditions for chaos. In fact snap-back repellers may be viewed as a special case of a fixed point with a transversal homoclinic orbits. Marotto (1979), furthermore, showed that, in certain circumstances, the existence of a snap-back repeller of a particular equation implies the existence of a transversal homoclinic orbit of a higher dimensional problem. Applying the theorem of Marotto (1979) to our dynamical system we can prove the following.

Theorem 3. *There are sufficiently large α such that the map (5) has a transversal homoclinic orbit for all $\beta < \epsilon$ for some $\epsilon > 0$.*

For the proof of Theorem 3 see Appendix A. Thus, the existence of the chaotic attractor in our model is proven mathematically.

Non-stationary Chaos

Similarly to the previous example it is clear from the local stability condition that an increase of the value β causes the flip bifurcation of the equilibrium and we observe an attracting 2-cycle. However, in this case the usual flip-cascade does not occur when β is increased. An increase in β gives cause to the divergence of the trajectory, and the series of p_t become irregular around the time trend. (See Fig. 3a). In our example, the non-stationary series p_t can be transformed to a stationary series by differencing once $\Delta p_t = p_t - p_{t-1}$. See Fig. 3b. As β is further increased we can observe flip sequences leading to chaotic dynamics of Δp_t, as it was shown in Fig. 4. The non-stationary chaos with the upward trend conforms exactly to *non-rational and speculative bubble*, because the continuous rise in the price is mainly caused by the strong nonlinearity of the chartist excess demand function.

Although it seems to be difficult to prove the existence of non-stationary chaos generated by the original map (Eq. 5) mathematically[4], it may be inferred by the following approximative method. It is clear from Fig. 5 that a phase plot $(\Delta p_t, \Delta p_{t-1})$ of a time path of the map (Eq. 5) yields only points along the nonlinear curve. Thus, this nonlinear process may be sufficiently approximated by a one-dimensional map

$$\Delta p_t = f(\Delta p_{t-1}). \tag{9}$$

The map (Eq. 9) has an unique equilibrium Δp^*. It is illustrated graphically that the equilibrium point is a snap-back repeller. By simply iterating the multi-valued inverse of f with initial point $\Delta p_0 = \Delta p^*$, we can find all pre-image points. Since there are two possible values for each Δp_k, the manner of choosing them was similar to that of the previous example: for p_{-1} the

[4]As far as I know the non-stationary chaos we consider here is new.

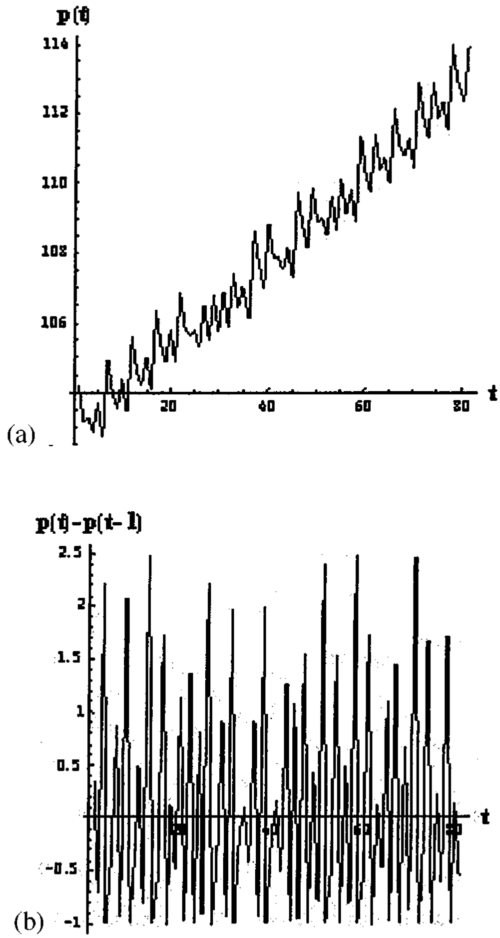

Fig. 3. (a) Time plots of the price p_t observed for the parameter values of $\alpha = 6$, $\beta = 6$, $\theta = 0.001$, $n = 1000$, $\kappa = 0.5$, and $\mu = 0.5$. (b) Time plots of the price change ($\Delta p_t = p_t - p_{t-1}$) observed for the parameter values of $\alpha = 6, \beta = 6, \theta = 0.001, n = 1000$, $\kappa = 0.5$, and $\mu = 0.5$.

root less than the value of Δp producing a maximum for $f(\Delta p)$ was selected, and for all $k \leq -1$, the greater root was chosen. Since the sequence is closer to the equilibrium point Δp^*, it is likely to find a snap-back repeller of f (see Fig. 5). Therefore we may say that the series of Δp_t is chaotic in the sense of Li and Yorke.

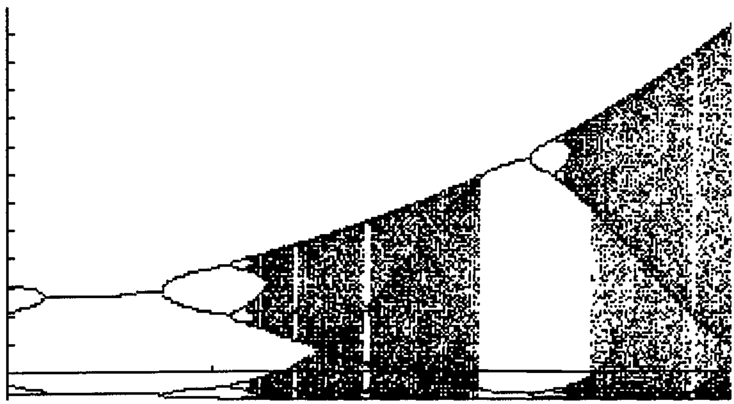

Fig. 4. The bifurcation diagram is a multiple valued plot of the attracting set of the price change ($\Delta p_t = p_t - p_{t-1}$) against the values of the parameter α for $\beta = 1, \theta = 0.001, n = 1000, \kappa = 0.5$, and $\mu = 0.5$.

CONCLUDING REMARKS

In this paper we propose a simple heterogeneous agent model of a financial market. To sum up the major results of our model, the strong nonlinearity of the fundamentalist excess demand function causes the stationary chaos around the fundamental value, while the strong nonlinearity of the chartist excess demand function leads to the non-stationary chaos or speculative bubbles, that is, irregular fluctuations of the asset price around a upward trend.

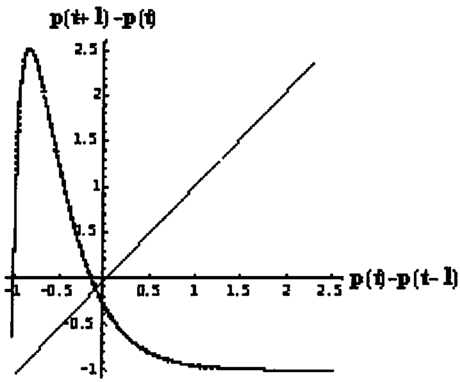

Fig. 5. Phase plot of a sequence of the price changes ($\Delta p_t = p_t - p_{t-1}$) observed for $\alpha = 6$, $\beta = 6, \theta = 0.001, n = 1000, \kappa = 0.5$, and $\mu = 0.5$.

APPENDIX A

Let us consider the following equation:

$$x_{k+1} = f(x_k, b_1 x_{k-1}, \ldots, b_m x_{k-m}). \tag{A1}$$

When $b_n = 0 (n = 1, \ldots, m)$ the equation (A1) reduces to the scalar equation:

$$x_{k+1} = f(x_k, 0, \ldots, 0). \tag{A2}$$

Assume that $b_i, x_k \in R$ and $f : R^{m+1} \to R$ be continuously differentiable. Then Marotto (1979) has proven the following:

Theorem (Marotto, 1979). *If (A2) has a snap-back repeller, then (A1) has a transversal homoclinic orbit for all $b_i < \epsilon$ for some ϵ.*

Let us now return to our model. The chartists' adaptive expectations can be rewritten as

$$p^e_{t+1} = \mu \sum_{k=0}^{t} (1-)^k p_{t-k}. \tag{A3}$$

Substituting the equation (A3) into the equation of the price adjustment, we get the equation

$$p_{t+1} = p_t + \theta n \Bigg[(1-\kappa) \exp(\alpha(p^* - p_t)) + \kappa \exp\left(\beta \left(\mu \sum_{k=0}^{t} (1-\mu)^k p_{t-k} - p_t \right) - 1 \right) \Bigg] \tag{A4}$$

Thus, if we can demonstrate that the problem:

$$p_{t+1} = p_t + \theta n (1-\kappa)(\exp(\alpha(p^* - p_t)) - 1), \tag{A5}$$

has a snap-back repeller, we can apply the above theorem to our dynamical system. We shall show this for appropriate values of α. We first need to find an interval $I = [p^* - r, p^* + r]$ with $|f'(p)| > 1$ for all $p \in I$ where $f(p) = p + \theta n(1-\kappa)(\exp(\alpha(p^* - p)) - 1)$. In particular, since $f'(p^*) < 0$ for $\alpha\theta n(1-\kappa) > 1$, we know that the left hand endpoint of I is arbitrary, i.e., $f'(p) < f'(p^*) < -1$ for $p < p^*$ and $\alpha\theta n(1-\kappa) > 1$. To estimate an acceptable right hand endpoint of I, note that $f'(p^* + \epsilon) = 1 - \alpha\theta n(1-\kappa)\exp(-\alpha\epsilon)$ and so, $f'(p^* + \epsilon) < -1$ for $\epsilon < \log(\alpha\theta n(1-\kappa)/2)/\alpha$. Thus, there exists positive values of ϵ such that $f'(p^* + \epsilon) < -1$ for $\alpha\theta n(1-\kappa) > 2$. If we restrict our discussion to values of $\alpha > 6$ for $\theta n(1-\kappa) = 0.5$, then $f'(p^* - \epsilon) < -1$ for all $\epsilon < 0.067$. Thus, letting $I = [p^* - 0.01, p^* + 0.01]$ is more than sufficient to insure that $f'(p) < -1$ for all $p \in I$, $\alpha > 6$ and $\theta n(1-\kappa) = 0.5$.

Now we must find a point $p_0 \in I$ with $F^M(p_0) = p^*$, $p_0 \neq p^*$ and $(F^M)'(p_0) \neq 0$ for some positive integer M. Since the multi-valued inverse of f cannot be written explicitly, pre-images of $p_0 = p^*$ can only be estimated numerically. Perhaps the simplest method to find p_0 satisfying the above conditions, therefore, is the following: with $p_0 = p^*$, we have two possible choices for each p_k, one greater than the p value producing a maximum for $f(p)$, that is, $p^* + \log(3)/6$ and one less. For $k = 0$ choosing the greater root will yield $p_{-1} = p^*$, which does not help in finding a p_0 with $p_0 \neq p^*$. Therefore, choose the greater root for $k = 0$. Now since we wish to find a pre-image point close to p^*, i.e., inside I, the optimal choice of roots is the smaller ones for all $k \leq -1$. If after M iterations with the selection of p_k's made as described, we find that $p_{-M} \in I$ and $f'(p_{-k}) \neq 0$ for $1 \leq k \leq M$, then letting $p_0 = p_{-M}$ will satisfy the hypotheses of Theorem 1. This is evident since: $f^M(p_0) = f^M(p_{-M}) = p_0 = p^*$. Also, $(f^M)'(p_0) = (f^M)'(p_{-M}) = \prod_{k=1}^{M} f'(p_{-k})$, and so, $f'(p_{-k}) \neq 0$ implies that $(f^M)'(p_0) \neq 0$.

A numerical study was performed upon the inverse function of $f(p)$ with $p_0 = p^*$ to find a point $p_0 = p_{-M}$ satisfying the above conditions. The

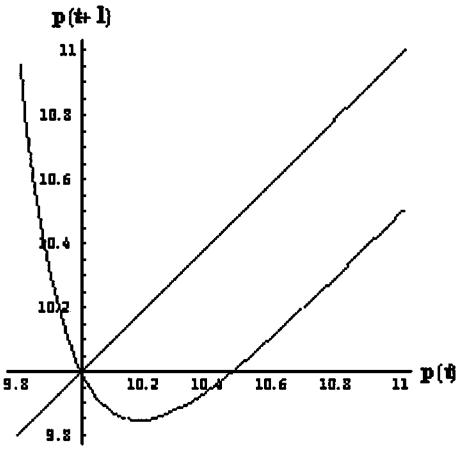

Fig. 6. Phase curve of the map $f(p) = p + \theta n(1-\kappa)(\exp(\alpha(p^* - p)) - 1)$ for $\alpha = 6$, $\theta n(1-\kappa) = 0.5$ and $p^* = 10$. If the parameter α is large enough, the requirements of the Marroto theorem are fulfilled, that is, the equilibrium point p^* is a snap-back repeller, as can be traced in this phase plot.

selection of p_k's was made in the manner described. The results show That for all values of $\alpha > 6$ a pre-image point of p^* lies within the interval I. Hence, p^* is a snap-back repeller and Theorem 2 guarantees chaos for all $\alpha > 6$ with $\theta n(1-\kappa) = 0.5$.

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