

# Speculative bubbles and fat tail phenomena in a heterogeneous agent model

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## Abstract

The aim of this paper is to propose a heterogeneous agent model of a stock market which leads to complicated endogenous price fluctuations. We find that a non-stationary chaos, that is considered as speculative bubble, is caused by the heterogeneity of traders' strategies. Furthermore, we show that the distributions of returns generated from the heterogeneous agent model have fat tails, which is a remarkable stylized fact observed in almost all financial markets.

**Key words:** heterogeneous interacting agents, non-stationary chaos, speculative bubbles, leptokurtosis, and fat tails.

## 1 Introduction

During the past decade, the interest in the dynamics of financial asset prices resulting from interaction of heterogeneous agents has been increasing rapidly. These price dynamics have been studied in e.g. Arthur et al. (1997), Brock and Hommes (1997, 1998), Chen and Yeh (2001), Chiarella (1992), Chiarella and He (1999), Chiarella et al. (2001), Day and Huang (1990), De Grauwe

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et al. (1993), Farmer (2000), Frankel and Froot (1988), Kirman (1991, 1993) and Lux (1995, 1998). An interesting approach is to show how heterogeneous interacting agent models actually generate the very statistical characteristics that are found empirically in financial time series, such as unpredictability of returns, fat tails, volatility clustering and long memory<sup>1</sup>. An extensive literature already exists (cf. Brock, Hommes and Wagener (2001), Gaunersdorfer and Hommes (2000), Kirman and Teyssiere (1998), LeBaron et al. (1999), Lux and Marchesi (1999, 2000), Youssefmir and Huberman (1997)). However, surprisingly few studies have so far tried to explain prolonged rises in asset prices, which are called *speculative bubbles*, due to interaction of heterogeneous agents with different trading strategies and expectations. In contrast, a large number both of theoretical and empirical studies has been made on rational bubbles, since Blanchard proposed this idea in 1979. The aim of this paper is to propose a heterogeneous-agent model that can generate speculative bubbles as a result of the interaction of heterogeneous agents. The theoretical set-up of the model presented here resembles that of previous works in many respects. We think of stock markets as consisting of a market maker who mediates the trading and generates the market-clearing price, and three typical types of traders that can be observed in financial practice<sup>2</sup>: *fundamentalists* who believe that the asset price will return to the fundamental price, *chartists* who base their trading decisions on an analysis of past price-trends, and *noise traders* who base their trading decisions on noisy information. Traders choose their trading strategy depending on the degree of accuracy of the price prediction. Our model differs from previous works in that the traders' excess demand functions for stock, which are derived from utility maximization, has a lower but no upper limit. We will see that this feature is the important factor that causes the speculative bubbles. In particular, we will derive the following results:

1. when the non-linearity of the excess demand functions is sufficiently strong, a speculative bubble or non-stationary chaos will be observed, that is, irregular fluctuations of the asset price around an upward price trend appear.
2. when speculative bubbles occur in the stock market, irrational traders who are chartists and noise traders will drive out of the market rational traders who are fundamentalists.

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<sup>1</sup>For a recent survey of the stylized facts on asset markets, see de Vries (1994), Pagan (1996) and Guillaume et al. (1997).

<sup>2</sup>These three different classes of traders are often used in the finance literature on the heterogeneous-agent systems.

3. a fat tailed distribution of the relative stock price change appears.

The structure of the paper is as follows: Chapter 2 presents the heterogeneous interacting agent model. Chapter 3 investigates the local and global properties of the deterministic dynamics in the model, and explains why speculative bubbles occur. Chapter 4 shows bubbles of stochastic dynamics and the fat tailed distribution of returns generated from the model. Chapter 5 gives brief concluding remarks.

## 2 The Model

Let us consider a simple stock market with the following characteristics. A large stock is traded. On the market, there are three typical groups of traders with different strategies, they are *fundamentalists*, *chartists* and *noise traders*. Traders can invest either in money or in stock. Since the model is designed to describe the stock price movements over short periods, such as one day, the dividend from stock and the interest rate for the risk-free asset will be omitted for simplicity. Traders are myopic and maximizing utility. Their utility depends on the price change they expect, and on their *excess demand* for stock rather than simply demand. Thus, their excess demand is derived from the utility maximization. What follows is a more precise account of the decision making of each trader type.

### 2.1 Fundamentalists

Let  $Y_t^f$  be the amount of money that a fundamentalist holds at time  $t$  and let  $X_t^f$  be the number of shares purchased by a fundamentalist at time  $t$ . Let  $p_t$  be the price per share of stock at time  $t$ . The fundamentalist's budget constraint is given by

$$Y_t^f + p_t X_t^f = Y_{t-1}^f + p_t X_{t-1}^f,$$

or equivalently

$$y_t^f + p_t x_t^f = 0, \tag{1}$$

where  $y_t^f = (Y_t^f - Y_{t-1}^f)$  denotes the fundamentalist's excess demand for money, and  $x_t^f = (X_t^f - X_{t-1}^f)$  his excess demand for stock. Suppose that the fundamentalist's preferences are represented by the utility function,

$$u = \alpha(y_t^f + p_{t+1}^f x_t^f) + \beta x_t^f - (1 + \beta x_t^f) \log(1 + \beta x_t^f), \quad \alpha > 0, \quad \beta > 0, \tag{2}$$

where  $p_{t+1}^f$  denotes the fundamentalist's expectation in period  $t$  about the price in the following period  $t + 1$ . The parameters  $\alpha$  and  $\beta$  are assumed to

be positive. Inserting (1) into (2) the fundamentalist's utility maximization problem becomes:

$$\max_{x_t^f} u = \alpha(p_{t+1}^f - p_t)x_t^f + \beta x_t^f - (1 + \beta x_t^f) \log(1 + \beta x_t^f). \quad (3)$$

The utility function  $u$  satisfies the standard properties:  $u'(|x^f|_t) > 0$ ,  $u''(|x^f|_t) < 0$  for all  $|x^f|_t \leq |x^{f*}|$  where  $|x^{f*}|$  denotes the absolute value of  $x^f$  producing a maximum utility. Thus, the utility function is strictly concave. It depends on the price change expected by fundamentalists ( $p_{t+1}^f - p_t$ ) as well as the fundamentalist's excess demand for stock  $x_t^f$ . The first part  $\alpha(p_{t+1}^f - p_t)x_t^f$  implies that a rise in the expected price change increases his utility. The remaining part expresses the attitude toward risk. Even if the expected price change is positive, he does not want to invest his total wealth in the stock, and vice versa. In this sense, fundamentalists are risk averse.  $\beta$  is the parameter that sets the lower limitation on excess demand. All excess demand for stock derived from the utility maximization is limited to  $-1/\beta$ . When the expected price change ( $p_{t+1}^f - p_t$ ) is positive, the maximum value of the utility function is also positive. This means that fundamentalists try to buy stock. Analogical, when the expected price change ( $p_{t+1}^f - p_t$ ) is negative, the maximum value of the utility function is negative, which means that they try to sell. The utility maximization problem (3) is solved for the fundamentalist's excess demand,

$$x_t^f = \frac{1}{\beta}(\exp(\frac{\alpha(p_{t+1}^f - p_t)}{\beta}) - 1). \quad (4)$$

Excess demand increases, as the expected price change ( $p_{t,t+1}^f - p_t$ ) increases. It has to be noticed that the optimal value of excess supply is limited to  $-1/\beta$  while the optimal value of excess demand is not restricted. Since there is little loss of generality in fixing the parameter  $\beta$  in unity, below, we will assume  $\beta$  to be constant and unity.

Then let us think of the fundamentalist's expectation formation. We assume that they form their price expectation according to a simple adaptive scheme:

$$p_{t+1}^f = p_t + \nu(p^* - p_t). \quad (5)$$

We see from Equation (5) that fundamentalists believe the price moving towards the fundamental price  $p^*$  by factor  $\nu$ . To sum up fundamentalist behavior: if the price  $p_t$  is below their expected price, they will try to buy stock, because they consider the stock to be undervalued. On the contrary,

if the price is above the expected value, they will try to sell, because they consider the stock to be overvalued.

## 2.2 Chartists

Chartists are assumed to have the same utility function as the fundamentalists (2). Their behavior is formalized as maximizing the utility function

$$v = \alpha(y_t + p_{t+1}^c x_t^c) + \beta x_t^c - (1 + \beta x_t^c) \log(1 + \beta x_t^c) \quad (6)$$

subject to the budget constraint

$$y_t^c + p_t x_t^c = 0 \quad (7)$$

where  $x_t^c$  and  $y_t^c$  represent the chartist's excess demand for stock and for money at period  $t$ , and  $p_{t+1}^c$  denotes the price expected by chartists. The chartist's excess demand function for the stock is given by

$$x_t^c = \frac{1}{\beta} (\exp(\frac{\alpha(p_{t+1}^e - p_t)}{\beta}) - 1). \quad (8)$$

Their expectation formation is as follows: They are assumed to forecast the future price  $p_{t+1}^c$  by using adaptive expectations,

$$p_{t+1}^c = p_t^c + \mu(p_t - p_t^c), \quad (9)$$

where the parameter  $\mu$  ( $0 < \mu < 1$ ) is a so-called error correction coefficient<sup>3</sup>. It follows that chartists try to buy stock when they anticipate a rising price for the next period, and, in contrast, try to sell stock when they expect a falling price.

## 2.3 Noise traders

Finally, let us consider the noise traders' decision making. They are assumed to decide on terms of noise in the sense of a large number of small events<sup>4</sup>. The behavior of a noise trader can be formalized as maximizing the quadratic utility function

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<sup>3</sup>The excess demand for chartists  $\exp(\beta(p_{t+1}^e - p_t)) - 1$  is rewritten as  $\exp(\beta(1 - \mu)(p_t^e - p_t)) - 1$  under adaptive expectations. Hence, chartists' decisions are based on observation of the past price-data. This type of trader, who simply extrapolates patterns of past prices, is a common stylized example, currently in popular use in heterogeneous agent models (cf. Frankel and Froot (1988) and Brock and Hommes (1997, 1998), and Gaunersdorfer and Hommes (2000)).

<sup>4</sup>For the meaning of noise in financial markets, see Black (1986).

$$W(x_t^n, y_t^n) = g(y_t^n + (p_t + \epsilon_t)x_t^n) - k(x_t^n)^2 \quad (10)$$

subject to the budget constraint

$$y_t^n + p_t x_t^n = 0$$

where  $x_t^n$  and  $y_t^n$  represent the noise trader's excess demand for stock and for money at time  $t$ , respectively. The noise  $\epsilon_t$  is assumed to be an IID random variable. The excess demand function for stock is given as

$$x_t^n = \gamma \epsilon_t, \quad \gamma = \frac{g}{2k} > 0 \quad (11)$$

where  $\gamma$  denotes the strength of reaction on noisy information. In short, noise traders try to buy stock, if they believe the noise to be good news ( $\epsilon_t > 0$ ). Inversely, if they believe the noise to be bad news ( $\epsilon_t < 0$ ), they try to sell it.

## 2.4 The adjustment process of the price

Let us leave traders' decision-making processes and turn to the adjustment of the stock-market price. We assume the existence of a *market maker*, such as a specialist in the New York stock exchange. The role of the market maker is to give an execution price to incoming orders and to execute transactions<sup>5</sup>.

He announces a price at the beginning of each trading period. Traders then determine their excess demand, based on the announced price and on their expected prices. When the market maker observes either excess demand or excess supply, he applies the so-called *short-side rule* to the demands and supplies, taking aggregate transactions for the stock to be equal to the minimum of total supply and demand. Thus traders on the short side of the market<sup>6</sup> will realize their desired transactions. At the beginning of the next trading period, he announces a new price. If the excess demand in period  $t$  is positive (negative), the market maker raises (reduces) the price for the following period  $t + 1$ . The process then is repeated. Let  $\kappa$  and  $\xi$  be the

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<sup>5</sup>We don't discuss about the market maker's own trading activity because to follow up this matter further would take us beyond the scope of this paper. The heterogeneous agent model that contains a specialist as a market participant is proposed by e.g. Ready (1999). Extension of the our model to allow for the market maker's own trade activity is a possible direction for future research.

<sup>6</sup>The short side of a market is that where the aggregate volume of desired transactions is smallest. It is thus the demand side if there is excess supply, the supply side if there is excess demand.

fractions of chartists and of noise traders in the total number of traders, respectively. Then the process of price adjustment can be written as

$$p_{t+1} - p_t = \theta n [(1 - \kappa - \xi)x_t^f + \kappa x_t^c + \xi x_t^n], \quad (12)$$

where  $\theta$  denotes the speed of the adjustment of the price, and  $n$  the total number of traders.

## 2.5 Strategy switching

An interesting question is whether the rational traders (fundamentalists) will drive irrational traders (chartists and noise traders) out of the market. As in many studies on heterogeneous interacting agent models (cf. Brock and Hommes (1997, 1998) and Lux (1998)), it may seem natural that switching between different trading strategies plays an important role. In this subsection, we ask the question how a trader makes his choice between fundamentalist and chartist strategy. The basic idea is that he chooses according to the accuracy of prediction. More precisely, we propose that the proportion of chartists  $\kappa_t$  is updated according to the difference between the squared prediction errors of each strategy. Formally, we write the dynamics of the proportion of chartists  $\kappa$  as

$$\kappa_t = \frac{(1 - \xi)}{1 + \exp(\psi(E_t^c - E_t^f))}, \quad (13)$$

$$E_t^c = (p_t - p_t^c)^2, \quad E_t^f = (p_t - p_t^f)^2,$$

where  $\psi$  measures how sensitive the mass of traders selects the optimal prediction strategy at period  $t$ . The parameter  $\psi$  is called the intensity of choice to switch trading strategies, according to Brock and Hommes (1997, 1998). Equation (13) expresses that if the chartists' squared prediction error  $E_t^c$  is smaller than the fundamentalists' one  $E_t^f$ , some fraction of fundamentalists will become chartists, and visa versa.

## 3 Deterministic Dynamics

The aim of this section is to describe the dynamical behavior of the model. In particular, we will indicate local and global properties of the deterministic dynamics when the parameter  $\alpha$  is allowed to vary.  $\alpha$  measures the strength of the non-linearity of the excess demand functions. Substituting (4), (5), (8), (9) and (11) to (12), the model can be rewritten as

$$p_{t+1} = p_t + \theta n [(1 - \kappa_t - \xi)(\exp(\alpha \nu (p^* - p_t)) - 1) + \kappa_t (\exp(\alpha (1 - \mu)(p_t^c - p_t)) - 1) + \xi \gamma \epsilon_t],$$

$$\begin{aligned} p_{t+1}^f &= p_t + \nu (p^* - p_t), \\ p_{t+1}^c &= p_t^c + \mu (p_t - p_t^c), \end{aligned} \tag{14}$$

where

$$\kappa_t = \frac{1}{1 + \exp(\psi((p_t - p_t^c)^2 - (p_t - p_t^f)^2))}.$$

The first of the equations above represents the adjustment of the price performed by the market maker. The second and third equations show the formation of the expectations of fundamentalists and chartists, respectively. The fourth equation represents the movement of the chartist fraction. Throughout this section, we assume that there exist no noise traders ( $\xi = 0$ ). Thus, the dynamics of the system (14) is deterministic. As can be checked easily, the dynamical system (14) has as an unique fixed point the point  $P \equiv (\bar{p}, \bar{p}^f, \bar{p}^c) = (p^*, p^*, p^*)$ .

### 3.1 The local stability conditions

To investigate the dynamics of the model, we shall first determine the local stability region of the unique equilibrium point  $P$ . The local stability analysis of the equilibrium point  $P$  is performed via evaluation of the three eigenvalues of the Jacobian matrix (14) at  $P$ . Let us denote by

$$c(\lambda) = \lambda^3 - T\lambda^2 + D\lambda \tag{15}$$

the associated characteristic polynomial of the Jacobian at  $P$ , where  $D = (2 - \mu - 0.5\theta n\alpha(1 - \mu + \nu))$  and  $T = (1 - \mu - 0.5\theta n\alpha(1 + \nu)(1 - \mu))$ . An eigenvalue of the Jacobian is 0, and the other roots  $\lambda_1, \lambda_2$  satisfy the relation

$$\lambda^2 - T\lambda + D = 0. \tag{16}$$

Thus, the stability of the equilibrium point is determined by the absolute values of  $\lambda_1$  and  $\lambda_2$ . The eigenvalues  $\lambda_1, \lambda_2$  are  $\lambda_{1,2} = T/2 \pm \sqrt{\Delta}/2$  where  $\Delta \equiv T^2 - 4D$ . As it is well known, a sufficient condition for the local stability consists of the following inequalities: (i)  $1 - T + D > 0$ , (ii)  $1 + T + D > 0$ , and (iii)  $D < 1$ , giving necessary and sufficient conditions for the two eigenvalues to be inside the unit circle of the complex plane. Elementary computations lead to the condition:

$$\alpha < \frac{3 - 2\mu}{\theta n(1 + \nu - \mu(1 + 0.5\nu))}. \quad (17)$$

The above local conditions demonstrate that while increasing  $\alpha$  starting from a sufficiently low value inside the stability region, a loss of local stability of the equilibrium point  $P$  may occur via a flip bifurcation, when crossing the curve

$$\alpha = \frac{3 - 2\mu}{n\theta(1 + \nu - \mu(1 + 0.5\nu))}. \quad (18)$$

### 3.2 Global bifurcations

What happens as  $\alpha$  is further increased? We will see that first, a stable 2-cycle appears, secondly, the time path of the price diverges, and thirdly, the time evolution of  $p_t$  displays a remarkable transition from regular to chaotic behavior around an upward time trend. It is important that the non-stationary chaos can be transformed to a stationary series by differentiation once  $\Delta p_t = p_t - p_{t-1}$ . The price series fluctuate irregularly around an upward price trend, and the series of the *price change* fluctuates chaotically within a finite interval. Below, this non-stationary chaos of the price will be referred to as a *speculative bubble*.

Let us look at these phenomena using the numerical simulation. In all numerical simulations below, we will use a set of parameters given by

$$\theta = 0.001, \quad n = 1000, \quad \mu = \nu = 0.5, \quad \gamma = 1, \quad p^* = 100, \quad \psi = 0.001. \quad (19)$$

Figure 1 shows a bifurcation diagram with the parameter  $\alpha$  as the bifurcation parameter. The figure summarizes the long-run time evolution of the price change  $\Delta p_t$ , for values of the parameter  $3 < \alpha < 7$ . By looking at Figure 1, it is evident that the dynamic behavior starts from a stable point, then goes through a sequence of period doubling bifurcations into a stable cycle of periods 2, 4, 8, 16, ..., and so on. Then, the strange attractor set occurs for values beyond the cycle of period three which appears at about  $\alpha = 6$ . This bifurcation scenario to chaos is the familiar route to chaos, that is to say, the *Feigenbaum scenario*.

(Figure 1 goes about here.)

### 3.3 Speculative bubbles

Why do speculative bubbles or non-stationary chaos appear when  $\alpha$  increases? Unfortunately, it is difficult to prove mathematically the existence of the strange attractor. However, as we shall see, it is possible to give an intuitive interpretation of speculative bubbles. To begin with, let us consider why a chaotic behavior of the price change is observed for large values of  $\alpha$ . Figure 2 indicates a phase plot  $(\Delta p_t, \Delta p_{t+1})$  of a sequence of the price change  $\Delta p_t$  for  $\alpha = 5.5$ . The phase plot is constructed by iteration of 20000 times, but here we use just the last 10000 iterations. A glance at Figure 2 shows a fixed point on the phase curve, depicted as an intersection between the phase curve and the 45-degree line<sup>7</sup>. As can be traced in this phase plot, a set of initial states of  $\Delta p_t$  close to the fixed point will be first stretched, but at some point the set will be folded back onto itself. This stretching and folding is the root of the chaotic price behavior (For mathematical proof see Li and York (1975), and Marotto (1978)).

(Figure 2 goes about here.)

The next question is why speculative bubbles occur when  $\alpha$  is large. The price adjustment equation suggests that a price trend can occur only when excess demand aggregated over two types of traders is positive on average. Why would average excess demand be positive as  $\alpha$  is large? For an answer, let us recall the traders' excess-demand functions (4) and (8).  $x_t$  denotes the traders' excess demand, and  $p_{t+1}^e$  denotes the expected price. The partial derivative of the excess-demand function with respect to the parameter  $\alpha$  is obtained as

$$\frac{\partial x_t}{\partial \alpha} = (p_{t+1}^e - p_t) \exp(\alpha(p_{t+1}^e - p_t)), \quad (20)$$

where we assume  $\beta = 1$ . This equation suggests a rise in  $\alpha$  increasing excess demand exponentially, given the expected price change  $(p_{t+1}^e - p_t)$  is positive. As shown in the preceding subsection, the maximum value of the excess demand increases freely when  $\alpha$  increases, while the maximum value of the excess supply never can be lower than  $-1/\beta$ , regardless of the value of  $\alpha$ . Assume that, in the initial state, the price  $p_t$  exceeds the fundamental price  $p^*$  in period  $t$ , and that the chartists forecast a rise over the next period. The fundamentalists predict a falling price. Then, chartists try to buy and fundamentalists try to sell stock. When  $\alpha$  is sufficiently large, one can safely state that chartist excess demand will exceed fundamentalist excess supply.

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<sup>7</sup>We can also show that the phase curve is stretched upwards when  $\alpha$  increases.

The market maker observes excess demand, and raises the price for the following period. If the rise is strong, chartists may predict a new rise, which provokes yet another reaction of the market maker. Once the price deviates strongly from the fundamental price, the fundamentalists are always sellers of stock, and when  $\alpha$  is sufficiently large, the chartists' excess demand may begin to exceed the fundamentalists' excess supply on average. Furthermore, fundamentalists may be driven out of the market by evolutionary competition expressed by (13)<sup>8</sup>. It follows that average excess demand is positive and speculative bubbles occur when the value of  $\alpha$  is sufficiently large.

## 4 Stochastic dynamics

To approach the model to real stock markets, we now assume the existence of noise traders. Below, we will put  $\xi = 0.3$ , i.e., 30 percent of the total number of traders are noise traders. Figure 3a and Figure 3b indicate the time series of the price and of the price change for  $\alpha = 5.5$ , disturbed by dynamic noise with  $\epsilon_t \sim N(0,1)$ . The figures show that the time series of the price has an upward trend and that it fluctuates irregularly<sup>9</sup>. (Figure 3a and Figure 3b go about here.)

### 4.1 Evolutionary competition

Another very interesting question is whether the attempts of the irrational traders (chartists and noise traders) can survive competition against the rational traders (fundamentalists). Figure 4a illustrates the time series of the fraction of chartists  $\kappa$  for the parameters  $\alpha = 4$  and  $\xi = 0.3$ . The speculative bubble does not occur under this set of parameters. Figure 9a shows that  $\kappa_t$  continues to fluctuate around 0.35. This means that, on average, the same number of fundamentalists and chartists exists, and that the irrational traders will stay in the market forever. On the other hand, Figure 4b shows the time series of the fraction of chartists  $\kappa_t$  for the parameters  $\alpha = 5$  and  $\xi = 0.3$ . There occurs a speculative bubble under this set of parameters. We see from Figure 9b that  $\kappa_t$  finally converges to 0.7. It means that fundamentalists die out, and irrational traders survive competition. The reasons are speculative bubbles caused by the strong non-linearity of the excess-demand

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<sup>8</sup>In the following section, we shall be examining this subject.

<sup>9</sup>Many numerical simulations for the various values of  $\alpha$  and  $\xi$  suggest that the existence of noise traders does not change essentially the properties of the price dynamics, such as speculative bubbles.

functions. When speculative bubbles occur, the price continues to deviate from the fundamental price, so that the prediction errors of the fundamentalists' trading strategy enlarge, comparing to the prediction errors of the chartists' trading strategy. Therefore, the number of fundamentalists is decreasing while the number of chartists is rising.

(Figure 4a and Figure 4b go about here.)

## 4.2 Fat tail phenomenon

A number of empirical studies have found that the empirical distributions of returns in almost all financial data show an uni-modal bell shape, and possess excessive fourth moments, so-called *leptokurtosis*, (cf. Pagan (1996)). Figure 5 is the histogram of the stock-return series <sup>10</sup> for  $\alpha = 4$ . It is constructed by 20000 iterations of the system (14) with  $\alpha = 5.5$ . The shape of the histogram appears well-behaved: an uni-modal bell-shape. Table 1 presents sample statistics of the stock returns, which are based upon several series of 20000 data points. The sixth and seventh columns denote the dynamical properties of the price and the price change, respectively. The mean value is very close to zero, and the standard deviation is small. Skewness is positive, and kurtosis is significantly positive. The important result is the histogram exhibiting apparently more probability mass in the tails and in the centre than the standard Normal. Therefore, the time series of the returns are leptokurtotic (excessive fourth moments).

(Figure 5 goes about here.)

Table 1: Statistics of return from artificial data

$\alpha$	<i>Mean</i>	<i>S.D.</i>	<i>Skewness</i>	<i>Kurtosis</i>	<i>Price</i>	<i>Return</i>	$\rho$
1	0	0.014	0.023	0.061	stable	stable	4.83
2	0	0.016	0.210	0.323	stable	stable	4.14
3	0	0.02	0.703	1.552	periodic	periodic	3.2
4	0	0.026	1.411	4.277	periodic	periodic	2.32
5	0	0.004	2.963	17.605	bubble	chaotic	4.58

However, the kurtosis is a relatively limited measure of deviation for Gaussian statistics. Recent literature finds that the distribution of returns is not

<sup>10</sup>Returns are defined as  $r_t = \log p_{t+1} - \log p_t$ .

only leptokurtotic, but also belongs to the class of *fat tailed* distributions. This finding is generally accepted as a universal characteristic of practically all financial returns. More formally, it has been shown that the tails of the distribution of returns follow approximately a power law:

$$F(|z_t| > x) \sim cx^{-\rho} \quad (21)$$

where the function  $F$  denotes the cumulative probability distribution of the normalized stock returns and  $\rho$  is called tail index. We normalize the stock return,

$$z_t = \frac{(r_t - \langle r_t \rangle)}{v} \quad (22)$$

where  $\langle r_t \rangle$  denotes the time average and  $v$  denotes the standard deviation. The last column of Table 1 shows the values of the tail index  $\rho$  for tail sizes up to 5 percent of the size of the return series generated from our model under various values of  $\alpha$ <sup>11</sup>. Most empirical studies show estimates of  $\rho$  falling in the range 2 to 4 (cf. Pagan (1996), Gopikrishnan et al. (1998)). In Table 1 the values of  $\rho$  range between 2 and 4, as long as periodic solutions of the price are observed. Thus, we find a power-law behavior for the cumulative distribution of the stock returns with  $\rho$  well outside the Levy regime ( $0 < \rho < 2$ ). Consequently, the time series of the returns generated from the above artificial stock market share the remarkable feature of actual data of stock markets.

## 5 Concluding Remarks

In this paper, we proposed a heterogeneous interacting agent model of a stock market. It generates three important findings: (1) the non-linearity of the excess demand functions derived from the traders' optimization behavior might generate speculative bubbles, (2) fundamentalists may be driven out of the market by irrational traders, such as chartists and noise traders, provided a speculative bubble occurs, and (3) the return distributions derived from the model have fat tails, what is an important characteristic of empirical financial data which is observed in almost all financial markets. There are, however, remarkable differences between the time series generated from our model and the actual financial data. For example, the noisy time series of the price change (Figure 9b) has strongly significant autocorrelations at the first two lags ( $\rho_1 = 0.5$  and  $\rho = 0.1$ ). Since in reality the series of

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<sup>11</sup>The values of  $\rho$  are computed by the Hill tail index estimate that is a standard work tool for estimations of the Pareto exponent of the tails.

price changes or returns have almost no significant autocorrelations at any lags, our estimation would be contradicting empirical series. Furthermore, the returns series generated from the model exhibit no volatility clustering, which is a universal property of real returns series. Apparently, our model is too simple to explain all stylized facts of real financial data at the same time and, thus, needs to be developed. This we leave to future work.

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## 7 Figure Captions

### Figure 1

Figure 1 is the bifurcation diagram of the stock return when the parameter  $\alpha$  varies smoothly from 3 into 7. The other parameter values:  $\theta = 0.001$ ,  $n = 1000$ ,  $\mu = \nu = 0.5$ ,  $\gamma = 1$ ,  $p^* = 100$ ,  $\psi = 0.001$ , and  $\xi = 0$ .

### Figure 2

Figure 2 indicates a phase plot  $(\Delta p_t, \Delta p_{t-1})$  of a sequence of the price change  $\Delta p_t$  generated for  $\alpha = 5.5$ . The other parameter values are:  $\theta = 0.001$ ,  $n = 1000$ ,  $\mu = \nu = 0.5$ ,  $\gamma = 1$ ,  $p^* = 100$ ,  $\psi = 0.001$ , and  $\xi = 0$ .

### Figure 3a

Figure 3a indicates the time series of the price for  $\alpha = 5.5$ , buffeted with dynamic noise with  $\epsilon \sim N(0, 1)$ . The other parameter values are:  $\theta = 0.001$ ,  $n = 1000$ ,  $\mu = \nu = 0.5$ ,  $\gamma = 1$ ,  $p^* = 100$ ,  $\psi = 0.001$ , and  $\xi = 0.3$ .

### Figure 3b

Figure 3b indicates the time series of the price change for  $\alpha = 5.5$ , buffeted with dynamic noise with  $\epsilon \sim N(0, 1)$ . The other parameter values are:  $\theta = 0.001$ ,  $n = 1000$ ,  $\mu = \nu = 0.5$ ,  $\gamma = 1$ ,  $p^* = 100$ ,  $\psi = 0.001$ , and  $\xi = 0.3$ .

### Figure 4a

Figure 4a illustrates the time-series plots of the fraction of the chartists  $\kappa$  for the parameters  $\alpha = 4$  and  $\xi = 0.2$ . The other parameter values are:  $\theta = 0.001$ ,  $n = 1000$ ,  $\mu = \nu = 0.5$ ,  $\gamma = 1$ ,  $p^* = 100$ ,  $\psi = 0.001$ , and  $\xi = 0.3$ .

### Figure 4b

Figure 4b illustrates the time series plots of the fraction of the chartists  $\kappa$  for the parameters  $\alpha = 5$  and  $\xi = 0.3$ . The other parameter values:  $\theta = 0.001$ ,  $n = 1000$ ,  $\mu = \nu = 0.5$ ,  $\gamma = 1$ ,  $p^* = 100$ ,  $\psi = 0.001$ , and  $\xi = 0.3$ .

### Figure 5

Figure 5 indicates the histogram of returns generated from the time series of the price for  $\alpha = 5.5$ . The other parameter values are:  $\theta = 0.001$ ,  $n = 1000$ ,  $\mu = \nu = 0.5$ ,  $\gamma = 1$ ,  $p^* = 100$ ,  $\psi = 0.001$ , and  $\xi = 0.3$ .