



Multiple equilibria and chaotic tâtonnement: Applications of the Yamaguti–Matano theorem

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Abstract

This paper demonstrates sufficient conditions for the discrete-time tâtonnement process to lead to chaos in the competitive economy with only two commodities. To prove that, we use Yamaguti–Matano theorem.

1. Introduction

Many important results on the stability of the competitive equilibrium have been obtained.¹ Most studies on the stability of the price adjustment process in a competitive economy (the tâtonnement process) have formulated the tâtonnement process by a differential equation system.² However, as pointed out in the literature the use of differential equations to describe the economic phenomenon of the tâtonnement process is a completely mathematical convention. For instance Saari (1985) describes the matter as: ‘It can be argued that the correct dynamical process associated with the tâtonnement process is an iterative one. Just one supporting argument is that the differential dynamic requires a continuum of information. At each instant of time the information must be updated; so a continuous case can be viewed as serving as a limiting approximation for the iterative dynamic. Thus it is only

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¹ See Negishi (1962) for an excellent survey.

² Exceptionally a formulation by a difference equation system was given in Uzawa (1960).

natural to expect the information requirements of the iterative dynamics to mimic those given by differential equations.’

In this context, Saari (1985) also showed that for any price adjustment rule which the economy follows, there exist excess demand functions where the discrete-time tâtonnement process fails to converge to an equilibrium. Furthermore, he suggested that even in the two commodity world, far more complex dynamic behavior can occur. This problem was approached by Day and Pianigiani (1991) and Bala and Majumdar (1992). They showed that there exist specific excess demand functions such that the discrete-time tâtonnement process with two commodities lead to chaos. However they did not demonstrate a sufficient condition for chaos in a discrete-time tâtonnement process.

Our purpose in this paper is the demonstration of a sufficient condition so that the discrete-time tâtonnement process leads to chaos in a competitive economy with two commodities.

To prove that we will use Yamaguti–Matano theorem [Yamaguti and Matano (1979)] to show that an asymptotically stable equilibrium point in the original differential equation is actually apt to turn into a source of chaos in the corresponding difference equation.

Our theorem (stated as Theorem 2 below) provides a sufficient condition for Li–Yorke Chaos in terms of (a) the sign pattern of excess demand function, and (b) the speed of adjustment. To put it more concretely, it is as follows: Given the following assumptions that excess demand must be negative for a sufficiently large price and that at least *two equilibria with nonnegative prices* exist in the competitive economy with two commodities and one of the equilibrium points is asymptotically stable, then there exists a positive constant speed of adjustment of the market such that the discrete-time tâtonnement process is chaotic in the sense of Li–Yorke [Li and Yorke (1975)] in an invariant finite interval.

The important point to notice is that it gives a lower bound for the speed over which the market tends to *overshoot* or *overreact*.

2. Model

Let us assume that in the economy there are two goods, commodity 1 and a numeraire which always has a price of 1. We also impose the following assumptions.

- (A-1) Excess demand must be negative for a sufficiently large price.
- (A-2) At least *two equilibria with nonnegative prices* exist in the competitive

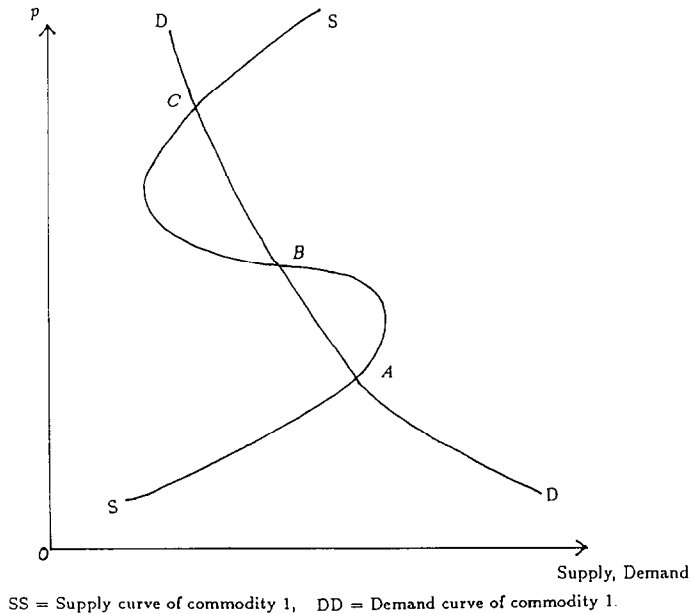


Fig. 1

economy with two commodities and one of the equilibrium points is asymptotically stable.³

A typical example of a curve that satisfies the assumptions just enumerated is the backward bending supply curves shown in Fig. 1.

Let us recall that Arrow and Hurwicz (1958) showed that the tâtonnement process in terms of the following differential equation :

$$dp/dt = g(p) \tag{1}$$

is *system stable* where $g(p)$ = excess demand function of commodity 1 at price p . In the case of Figure 1, A and C are asymptotically stable while B is unstable.

Up to this point a price adjustment process has been expressed as the

³ As an example that the above assumptions do not hold, let us consider the famous Gale-type example of globally instable equilibrium positions where there are three equilibria, only one having strictly positive prices for *both* goods and being unstable, the other two with a zero price for one good. [See Gale (1963).] It is clear that the example doesn't satisfy (A-1). Thus the result presented below cannot be applicable to the case.

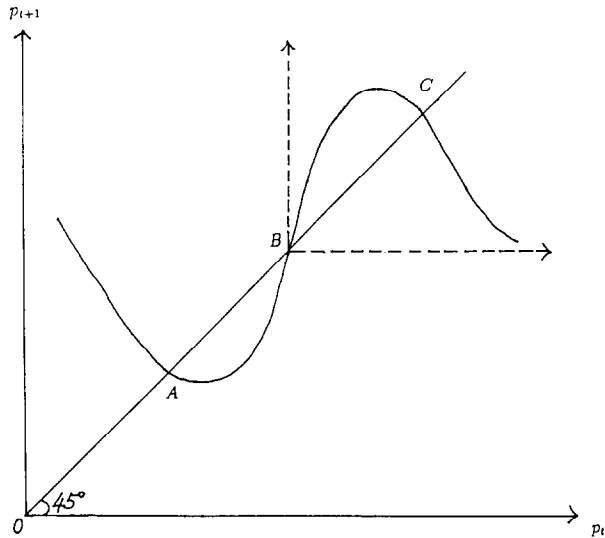


Fig. 2

scalar ordinary differential equation (1). However it should be emphasized here that lags exist in the price adjustment process in the real market. The lags can arise for the following reasons:

A price adjustment process is based upon a reaction of the people in the market as reflected by the disequilibrium of supply and demand. That is, people need to obtain information on excess demand, and this may be more or less out of date when the records are in. Decisions have then to be made and put into effect, with more delay.

Therefore, let us consider the following iterative price mechanism formulated by Saari (1985):

$$p_{t+1} = p_t + \alpha g(p_t) = G_\alpha(p_t), \tag{2}$$

where α is the speed of adjustment of the market and is constant non-negative.

Equation (2) is Euler's difference scheme for (1). Now consider the largest and the second largest of the equilibrium prices (in Figure 2, points C and B, respectively).

After shifting the origin as shown (by the dotted line) in Figure 2, the above assumptions (A-1) and (A-2) reduce to the conditions below:

$$\begin{aligned} g(0) &= g(\bar{p}) = 0, \\ g(p) &> 0 \quad (0 < p < \bar{p}), \\ g(p) &> 0 \quad (\bar{p} < p < +\infty), \end{aligned} \tag{3}$$

where \bar{p} denotes the largest of the equilibrium price and 0 the second largest of the equilibrium price.

Given these conditions, we can already apply Yamaguti–Matano theorem [Yamaguti and Matano (1979)] to the price adjustment process in a competitive economy. Their theorem is as follows:

Theorem 1 [Yamaguti and Matano (1979)]. Let us consider the scalar differential equations of the form

$$du/dt = f(u), \tag{4}$$

where $f(u)$ is continuous in R^1 . We assume that (4) has at least two equilibrium points one of which is asymptotically stable.⁴

As is easily seen, this assumption reduces (after a linear transformation of the unknown if necessary) to the conditions

$$\begin{aligned} f(0) &= f(\bar{u}) = 0 \text{ (for some } \bar{u} > 0), \\ f(u) &> 0 \text{ (} 0 < u < \bar{u}), \\ f(u) &< 0 \text{ (} \bar{u} < u < K). \end{aligned} \tag{5}$$

Here the constant K is possibly $+\infty$. Euler’s difference scheme for (4) takes the form

$$x_{t+1} = x_t + \Delta t f(x_t), \tag{6}$$

and henceforth we will adopt the notation $F_{\Delta t}(x) = x + \Delta t f(x)$. Our theorem can now be stated as follows:

(i) Let (5) hold. Then there exists a positive constant c_1 such that for any $\Delta t > c_1$ the difference equation (6) is chaotic in the sense of Li–Yorke.

(ii) Suppose in addition that $K = +\infty$; then there exists another constant c_2 , ($0 < c_1 < c_2$), such that for any $0 \leq \Delta t \leq c_2$, the map $F_{\Delta t}$ has an invariant finite interval $[0, \varphi_{\Delta t}]$ (i.e., $F_{\Delta t}$ maps $[0, \varphi_{\Delta t}]$ into itself) with $\varphi_{\Delta t} > \bar{u}$. Moreover, when $c_1 < \Delta t \leq c_2$, the above-mentioned chaotic phenomenon occurs in this invariant interval.

Remark. It is not difficult to see that if $f(x)$ is analytic and (4) has no

⁴ The mathematical definition of asymptotically stable [see Hirsch and Smale (1974, Ch. 9)] is as follows. Suppose $\bar{u} \in W$ is an equilibrium of the differential equation (4) where $f: W \rightarrow E$ is a C^1 map from an open set W of the vector space E into E . Then a fixed point \bar{u} is asymptotically stable if every neighborhood $U \subset W$ of \bar{u} contains a neighborhood U_1 of \bar{u} such that every solution $u(t)$ with $u(0)$ in U_1 is defined and in U for all $t \geq 0$, and $\lim_{t \rightarrow +\infty} u(t) = \bar{u}$.

asymptotically stable equilibrium point, then (6) can never be chaotic for any nonnegative value of Δt .

Using this theorem we can formulate the following:

Theorem 2. Let the above conditions (3) hold. Then there exist a positive constant α_1 and α_2 , ($0 < \alpha_1 < \alpha_2$), such that for any $0 \leq \alpha \leq \alpha_2$ the map G_α has an invariant finite interval $[0, \varphi_\alpha]$ (i.e., G_α maps $[0, \varphi_\alpha]$ into itself) with $\varphi_\alpha > \bar{p}$. Moreover, when $\alpha_1 < \alpha \leq \alpha_2$, the difference equation (2) is chaotic in the sense of Li–Yorke in this invariant finite interval.

The proof of Theorem 2 is trivial once we have Theorem 1 (Y–M theorem).

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