Quiz 1

ID#: **Name:**

Let p, q and r be statements, and x, y and z compound statements of p, q and r. Let

$$s \equiv (p \Rightarrow r) \wedge (q \Rightarrow r), \quad t \equiv (p \vee q) \Rightarrow r.$$

1. Complete the truth table of s and t. Clarify the columns of the truth values of the statements.

p	q	r	$(p \Rightarrow r) \wedge (q \Rightarrow r)$	$(p \vee q) \Rightarrow r$	x	y	z
T	T	T			F	F	F
T	T	F			F	T	T
T	F	T			F	F	T
T	F	F			F	F	F
F	T	T			F	F	F
F	T	F			F	F	F
F	F	T			F	F	F
F	F	F			T	T	T

2. Choose the correct one.

   (a) $s \equiv t$ (b) $s \equiv \neg t$ (c) Neither (a) nor (b).

3. Express $s \equiv (p \Rightarrow r) \wedge (q \Rightarrow r)$ using $\neg$, $\vee$ and parentheses only. Show work or give a brief explanation.

4. Fill each underlined blank with $\neg$, $\wedge$ or $\vee$ to express x, y and z in the truth table above. There may be voids.

$$\begin{aligned} x &\equiv ((\_\_\ p)\ \_\_\ (\_\_\ q))\ \_\_\ (\_\_\ r), \\ y &\equiv x \vee (((\_\_\ p)\ \_\_\ (\_\_\ q))\ \_\_\ (\_\_\ r)), \\ z &\equiv y\ \_\_(((\_\_\ p)\ \wedge\ (\_\_\ q))\ \_\_\ (\_\_\ r)). \end{aligned}$$

Message: What is your dream? Describe your vision of yourself and the world 25 years from now. 将来の夢、25 年後の自分について、世界について。(If you don't want your message to be posted, write "Do Not Post."「HP 掲載不可」は明記の事。)

Solutions to Quiz 1

Let p, q and r be statements, and x, y and z compound statements of p, q and r. Let

$$s \equiv (p \Rightarrow r) \wedge (q \Rightarrow r), \quad t \equiv (p \vee q) \Rightarrow r.$$

1. Complete the truth table of s and t. Clarify the columns of the truth values of the statements.

p	q	r	$(p \Rightarrow r)$	$\wedge$	$(q \Rightarrow r)$	$(p \vee q)$	$\Rightarrow$	r	x	y	z
T	T	T		***T***		T	***T***	T	F	F	F
T	T	F	F	***F***	F	T	***F***	F	F	T	T
T	F	T		***T***		T	***T***	T	F	F	T
T	F	F	F	***F***		T	***F***	F	F	F	F
F	T	T		***T***		T	***T***	T	F	F	F
F	T	F		***F***	F	T	***F***	F	F	F	F
F	F	T		***T***		F	***T***	T	F	F	F
F	F	F		***T***		F	***T***	F	T	T	T

Solutions are in bold face.

2. Choose the correct one.

$\boxed{\text{(a) } s \equiv t}$ (b) $s \equiv \neg t$ (c) Neither (a) nor (b).

3. Express $s \equiv (p \Rightarrow r) \wedge (q \Rightarrow r)$using $\neg$, $\vee$ and parentheses only. Show work or give a brief explanation.

Solution.

$$\begin{aligned} s &\equiv (p \Rightarrow r) \wedge (q \Rightarrow r) \\ &\equiv (\neg p \vee r) \wedge (\neg q \vee r) \qquad \text{as } x \Rightarrow y \equiv \neg x \vee y \\ &\equiv \neg(\neg(\neg p \vee r) \vee \neg(\neg q \vee r)) \qquad \text{as } x \wedge y \equiv \neg(\neg x \vee \neg y). \end{aligned}$$

The Second Solution. In the following, the first step is same as above, and then the distributive law and the de Morgan's rule are applied.

$$s \equiv (p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (\neg p \vee r) \wedge (\neg q \vee r) \equiv (\neg p \wedge \neg q) \vee r \equiv \neg(p \vee q) \vee r.$$

The Third Solution. Since $s \equiv t$, we have

$$s \equiv t \equiv (p \vee q) \Rightarrow r \equiv \neg(p \vee q) \vee r.$$

4. Fill each underlined blank with $\neg$, $\wedge$ or $\vee$ to express x, y and z in the truth table above. There may be voids.

$$\begin{aligned} x &\equiv ((\underline{\neg}\, p) \,\underline{\wedge}\, (\underline{\neg}\, q)) \,\underline{\wedge}\, (\underline{\neg}\, r), \\ y &\equiv x \vee (((\underline{}\, p) \,\underline{\wedge}\, (\underline{}\, q)) \,\underline{\wedge}\, (\underline{\neg}\, r)), \\ z &\equiv y \,\underline{\vee}(((\underline{}\, p) \ \wedge\ (\underline{\neg}\, q)) \,\underline{\wedge}\, (\underline{}\, r)). \end{aligned}$$

Quiz 2

ID#: **Name:**

Let $[i,j;c],[i,j],[i;c]$ be the following elementary row operations. (1) $[i,j;c]$: Replace row i by the sum of row i and c times row j. (2) $[i,j]$: Interchange row i and row j. (3) $[i;c]$: Multiply all entries in row i by a nonzero constant c.

We applied elementary row operations to the augmented matrix A of a system of linear equations.

$$A = \begin{bmatrix} 0 & 3 & 15 & 0 & -3 & 0 & 24 \\ 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 1 & 1 & -1 & 0 & 0 & -8 & 2 \end{bmatrix} \xrightarrow{(a)} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 3 & 15 & 0 & -3 & 0 & 24 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 1 & 1 & -1 & 0 & 0 & -8 & 2 \end{bmatrix} \xrightarrow{(b)}$$

$$\begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 3 & 15 & 0 & -3 & 0 & 24 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{(c)} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 1 & 5 & 0 & -1 & 0 & 8 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 \end{bmatrix}$$

1. Write elementary row operations at (a), (b), (c) in the form $[i,j;c],[i,j]$, or $[i;c]$. (For example, (a) $[1,2;3]$, (b) $[2,3]$, (c) $[4;-1/2]$.)

   (a) $\boxed{}$ (b) $\boxed{}$ (c) $\boxed{}$

2. Find the reduced row echelon form of A by applying elementary row operations to the fourth matrix above. <u>Show work!</u>

3. Find the rank of the augmented matrix A of the system of linear equations, and the rank of the coefficient matrix C.

   (a) rank $A =$ (b) rank $C =$

4. Find the solutions assuming that $x_1, x_2, x_3, x_4, x_5, x_6$ are the unknowns.

Message: What is most precious to you? あなたにとって一番たいせつな（または、たいせつにしたい）もの、ことはなんですか。(If you don't want your message to be posted, write "Do Not Post."「HP 掲載不可」は明記の事。)

Solutions to Quiz 2

Let $[i,j;c]$, $[i,j]$, $[i;c]$ be the following elementary row operations. (1) $[i,j;c]$: Replace row i by the sum of row i and c times row j. (2) $[i,j]$: Interchange row i and row j. (3) $[i;c]$: Multiply all entries in row i by a nonzero constant c.

We applied elementary row operations to the augmented matrix A of a system of linear equations.

$$A = \begin{bmatrix} 0 & 3 & 15 & 0 & -3 & 0 & 24 \\ 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 1 & 1 & -1 & 0 & 0 & -8 & 2 \end{bmatrix} \xrightarrow{(a)} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 3 & 15 & 0 & -3 & 0 & 24 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 1 & 1 & -1 & 0 & 0 & -8 & 2 \end{bmatrix} \xrightarrow{(b)}$$

$$\begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 3 & 15 & 0 & -3 & 0 & 24 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{(c)} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 1 & 5 & 0 & -1 & 0 & 8 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 \end{bmatrix}$$

1. Write elementary row operations at (a), (b), (c) in the form $[i,j;c]$, $[i,j]$, or $[i;c]$. (For example, (a) $[1,2;3]$, (b) $[2,3]$, (c) $[4;-1/2]$.)

   (a) [1,2] (b) [4,1;-1] (c) [2;1/3]

2. Find the reduced row echelon form of A by applying elementary row operations to the fourth matrix above. Show work!

$$\begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 1 & 5 & 0 & -1 & 0 & 8 \\ 0 & -2 & -10 & 1 & 2 & -2 & -17 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{[3,2;2]} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 1 & 5 & 0 & -1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{[4,2;-1]}$$

$$\begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 1 & 5 & 0 & -1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -8 \end{bmatrix} \xrightarrow{[2,4;1]} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 & -9 & 2 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -8 \end{bmatrix}$$

   The third matrix is not in reduced row echelon form as it does not satisfy Condition 4.

3. Find the rank of the augmented matrix A of the system of linear equations, and the rank of the coefficient matrix C.

   Solution. The rank of a matrix is the number of nonzero rows in the reduced row echelon form of the matrix. The coefficient matrix is the matrix obtained from the augmented matrix by eliminating the last column.

   (a) rank $A = 4$ (b) rank $C = 4$

4. Find the solutions assuming that $x_1, x_2, x_3, x_4, x_5, x_6$ are the unknowns.

   Solution. Since rank $A =$ rank $C = 4$, this system is consistent. In this case, there are 6 unknowns, we need $2 = 6 - 4 = 6 -$ rank A free parameters. Since the third and the sixth column in the reduced row echelon form of the coefficient matrix do not have leading one's, we set $x_3 = s, x_6 = t$ to be free parameters.

$$\left\{ \begin{aligned} x_1 - 6x_3 - 9x_6 &= 2 \\ x_2 + 5x_3 + x_6 &= 0 \\ x_4 - 2x_6 &= -1 \\ x_5 + x_6 &= -8, \end{aligned} \right. \quad \text{yields} \quad \left\{ \begin{aligned} x_1 &= 6s + 9t + 2 \\ x_2 &= -5s - t \\ x_3 &= s \\ x_4 &= 2t - 1 \\ x_5 &= -t - 8 \\ x_6 &= t \end{aligned} \right.$$

   If the matrix is in reduced row echelon form and parameters are taken properly, $x_1, x_2, \ldots, x_6$ can be expressed by the sum of a constant and scalar multiples of parameters.

Quiz 3

ID#: **Name:**

$$A = \begin{bmatrix} 3 & -2 & -6 \\ -2 & 1 & 4 \\ -2 & 2 & 5 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix}, \ C = \begin{bmatrix} 3 & -2 & -6 & 1 & 0 & 0 \\ -2 & 1 & 4 & 0 & 1 & 0 \\ -2 & 2 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$ and C be as above, where $C = [A, I]$. We will find the inverse of A.

$$C \to C_1 = \begin{bmatrix} 3 & -2 & -6 & 1 & 0 & 0 \\ 1 & -1 & -2 & 1 & 1 & 0 \\ -2 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \to C_2 \to C_3 = \begin{bmatrix} 1 & -1 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ -2 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \to \cdots$$

The matrices C_1, C_2 and C_3 are obtained from C, C_1 and C_2 respectively by applying an elementary row operation once. Let S, T and U be 3×3 elementary matrices satisfying $SC = C_1$, $TC_1 = C_2$ and $UC_2 = C_3$.

1. Find both S and the inverse of S.

2. Find the product UT of matrices U and T. (Not TU!)

3. Find the inverse A^{-1} of A.

4. Suppose $A\boldsymbol{x} = \boldsymbol{b}$. Find x, y, z using the inverse of A.

Messages: Anything that made you rejoice, sad or angry, or you are thankful for recently? 最近とても嬉しかった（感謝している）こと、悲しかったこと、怒っていること。(If you don't want your message to be posted, write "Do Not Post."「ホームページ掲載不可」は明記のこと。)

Solutions to Quiz 3

$$A = \begin{bmatrix} 3 & -2 & -6 \\ -2 & 1 & 4 \\ -2 & 2 & 5 \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix},\ C = \begin{bmatrix} 3 & -2 & -6 & 1 & 0 & 0 \\ -2 & 1 & 4 & 0 & 1 & 0 \\ -2 & 2 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$ and C be as above, where $C = [A, I]$. We will find the inverse of A.

$$C \to C_1 = \begin{bmatrix} 3 & -2 & -6 & 1 & 0 & 0 \\ 1 & -1 & -2 & 1 & 1 & 0 \\ -2 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \to C_2 \to C_3 = \begin{bmatrix} 1 & -1 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ -2 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \to \cdots$$

The matrices C_1, C_2 and C_3 are obtained from C, C_1 and C_2 respectively by applying an elementary row operation once. Let S, T and U be 3×3 elementary matrices satisfying $SC = C_1$, $TC_1 = C_2$ and $UC_2 = C_3$.

Solution. The elementary row operations applied are $[2,1;1]$, $[1,2]$ and $[2,1;-3]$. The second and the third can be $[1,2;-3]$ and $[1,2]$. Therefore, $S = E(2,1;1)$ and $T = E(1,2)$ and $U = E(2,1;3)$. (Or, $T = E(1,2;-3)$ and $U = E(1,2)$.)

1. Find both S and the inverse of S.

   Solution.

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, S^{-1} = E(2,1;1)^{-1} = E(2,1;-1) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \text{or}$$

$$[S, I] = [E(2,1;1), I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = [I, E(2,1;-1)].$$

2. Find the product UT of matrices U and T. (Not TU!)

   Solution.

$$UT = E(2,1;-3)E(1,2) = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E(1,2)E(1,2;-3).$$

   Note that the UT is the same for both cases.

3. Find the inverse A^{-1} of A.

   Solution. $C \to C_3 \to$

$$\begin{bmatrix} 1 & -1 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -1 & -2 & 0 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 & 2 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} = [I, A^{-1}]. \quad \text{Hence, } A^{-1} = \begin{bmatrix} 3 & 2 & 2 \\ -2 & -3 & 0 \\ 2 & 2 & 1 \end{bmatrix}.$$

   Corresponding elementary row operations applied to C_3 are $[3,1;2], [1,2;1], [1,3;2]$ in this order.

4. Suppose $A\boldsymbol{x} = \boldsymbol{b}$. Find x, y, z using the inverse of A.

   Solution. Since $\boldsymbol{x} = I\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$, by multiplying A^{-1} to $\boldsymbol{b}$, we have the following.

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 3 & 2 & 2 \\ -2 & -3 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \left\{ \begin{array}{rcl} x & = & 1, \\ y & = & 1, \\ z & = & 1. \end{array} \right.$$

Quiz 4

ID#: **Name:**

Let $f(x) = x^4 - 6x^3 + 12x^2 - 9x - 7$.

1. Find a polynomial $q(x)$ and a number r satisfying $f(x) = q(x)(x-2) + r$. Show work.

2. Find a, b, c, d, e satisfying $f(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 + e(x-2)^4$. Show work.

3. Find a polynomial $Q(x)$ of degree 3 such that $Q(-2) = 0$, $Q(-1) = 0$, $Q(0) = 0$, $Q(1) = 1$.

4. Let $h(x) = a(x+1)x(x-1) + b(x+2)x(x-1) + c(x+2)(x+1)(x-1) + d(x+2)(x+1)x$. Find a, b, c, d when $h(-2) = 12$, $h(-1) = 4$, $h(0) = -6$, $h(1) = 12$. Show work.

5. Find a polynomial $g(x)$ of degree 5 satisfying $g(-2) = 12$, $g(-1) = 4$, $g(0) = -6$, $g(1) = 12$ and $g(2) = h(2)$, where $h(x)$ is in 4.

Messages: What kind of adult is admirable? What is admirable about children? どんなおとなが魅力的ですか。こどもの魅力は何でしょう。(If you don't want your message to be posted, write " Do Not Post. " 「HP 掲載不可」は明記の事。)

Solutions to Quiz 4

Let $f(x) = x^4 - 6x^3 + 12x^2 - 9x - 7$.

1. Find a polynomial $q(x)$ and a number r satisfying $f(x) = q(x)(x-2) + r$. Show work.

   Solution. Applying synthetic division

$$\begin{array}{c|ccccc}
2 & 1 & -6 & 12 & -9 & -7 \\
 & & 2 & -8 & 8 & -2 \\
\hline
 & 1\,(c_3) & -4\,(c_2) & 4\,(c_1) & -1\,(c_0) & \boxed{-9\,(r)}
\end{array}$$

$$q(x) = c_3x^3 + c_2x^2 + c_1x + c_0 = x^3 - 4x^2 + 4x - 1, \qquad r = -9.$$

2. Find a, b, c, d, e satisfying $f(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 + e(x-2)^4$. Show work.

   Solution. By synthetic division below, $a = r = -9, b = -1, c = 0, d = 2, e = 1$.

$$\begin{array}{c|cccc}
2 & 1 & -4 & 4 & -1 \\
 & & 2 & -4 & 0 \\
\hline
2 & 1 & -2 & 0 & \boxed{-1\,(b)} \\
 & & 2 & 0 & \\
\hline
2 & 1 & 0 & \boxed{0\,(c)} & \\
 & & 2 & & \\
\hline
 & 1\,(e) & \boxed{2\,(d)} & &
\end{array}$$

$$\begin{aligned}
f(x) &= x^4 - 6x^3 + 12x^2 - 9x - 7 = (x^3 - 4x^2 + 4x - 1)(x-2) - 9 \\
&= ((x^2 - 2x)(x-2) - 1)(x-2) - 9 = ((x(x-2) + 0)(x-2) - 1)(x-2) - 9 \\
&= (((x-2) + 2)(x-2) + 0)(x-2) - 1)(x-2) - 9 \\
&= -9 - (x-2) + 0(x-2)^2 + 2(x-2)^3 + (x-2)^4.
\end{aligned}$$

3. Find a polynomial $Q(x)$ of degree 3 such that $Q(-2) = 0$, $Q(-1) = 0$, $Q(0) = 0$, $Q(1) = 1$.

   Solution.

$$Q(x) = \frac{(x+2)(x+1)x}{(1+2)(1+1)(1)} \left(= \frac{1}{6}(x+2)(x+1)x = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x \right).$$

4. Let $h(x) = a(x+1)x(x-1) + b(x+2)x(x-1) + c(x+2)(x+1)(x-1) + d(x+2)(x+1)x$. Find a, b, c, d when $h(-2) = 12$, $h(-1) = 4$, $h(0) = -6$, $h(1) = 12$. Show work.

   Solution.
$$\begin{aligned}
12 &= h(-2) = a(-2+1)(-2)(-2-1) = -6a, \quad a = -2, \\
4 &= h(-1) = b(-1+2)(-1)(-1-1) = 2b, \quad b = 2, \\
-6 &= h(0) = c(0+2)(0+1)(0-1) = -2c, \quad c = 3 \\
12 &= h(1) = d(1+2)(1+1)(1) = 6d, \quad d = 2
\end{aligned}$$

5. Find a polynomial $g(x)$ of degree 5 satisfying $g(-2) = 12$, $g(-1) = 4$, $g(0) = -6$, $g(1) = 12$ and $g(2) = h(2)$, where $h(x)$ is in 4.

   Solution.

$$\begin{aligned}
g(x) &= h(x) + (x+2)(x+1)x(x-1)(x-2) \\
&= (x+2)(x+1)x(x-1)(x-2) - 2(x+1)x(x-1) \\
&\quad 2(x+2)x(x-1) + 3(x+2)(x+1)(x-1) + 2(x+2)(x+1)x. \\
(&= x^5 + 14x^2 + 3x - 6)
\end{aligned}$$

   You do not need to expand in 3 or 5.

Quiz 5

ID#: **Name:**

1. Find the limit of the following if it exists, and write divergent, otherwise. Show work!

   (a) $\displaystyle\lim_{n\to\infty} \frac{5^n}{(-3)^n}$

   (b) $\displaystyle\lim_{n\to\infty} \frac{1-n+3n^2-5n^3}{1+n^3}$

   (c) $\displaystyle\lim_{n\to\infty} \frac{1-n+3n^2}{1+n^3}$

   (d) $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

   (e) $\displaystyle\lim_{x\to 2} \frac{x^2-x-6}{x-2}$

   (f) $\displaystyle\lim_{x\to -3} \frac{x^2+x-6}{x-2}$

2. Find the limit of the following. Show work.
   $\displaystyle\lim_{x\to 2} \frac{2x^3-11x^2+20x-12}{x^3-x^2-8x+12}$

3. Let $f(x) = q(x)(x-2)+3$, where $q(x)$ is a polynomial. Suppose $q(3) = -5$. Explain that there is c $(2 < c < 3)$ such that $f(c) = 0$.

Message: ICU aims to nurture trustworthy global citizen. What do you think is the key? ICU は「信頼される地球市民を育む」ことを目指していますが、鍵は何だと思いますか。(If you don't want your message to be posted, write " Do Not Post. "「ホームページ掲載不可」の場合は明記のこと。)

Solutions to Quiz 5

1. Find the limit of the following if it exists, and write divergent, otherwise. Show work!

   (a) $\displaystyle\lim_{n\to\infty} \frac{5^n}{(-3)^n} = \lim_{n\to\infty} \left(\frac{5}{-3}\right)^n$: divergent, as $|-5/3| > 1$. (The limit does not exist.)

   (b) $\displaystyle\lim_{n\to\infty} \frac{1-n+3n^2-5n^3}{1+n^3} = \lim_{n\to\infty} \frac{\frac{1}{n^3}-\frac{1}{n^2}+\frac{3}{n}-5}{\frac{1}{n^3}+1} = -5.$

   (c) $\displaystyle\lim_{n\to\infty} \frac{1-n+3n^2}{1+n^3} = \lim_{n\to\infty} \frac{\frac{1}{n^3}-\frac{1}{n^2}+\frac{3}{n}}{1+\frac{1}{n^3}} = 0.$

   (d) $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x+3)(x-2)}{x-2} = \lim_{x\to 2} x+3 = 5.$

   (e) $\displaystyle\lim_{x\to 2} \frac{x^2-x-6}{x-2} = \lim_{x\to 2} \frac{(x+1)(x-2)-4}{x-2} \sim \frac{-4}{0}$: divergent. (The limit does not exist.)

   (f) $\displaystyle\lim_{x\to -3} \frac{x^2+x-6}{x-2} = \lim_{x\to -3} \frac{(x+3)(x-2)}{x-2} = \frac{0}{-5} = 0.$

2. Find the limit of the following. Show work.
   $\displaystyle\lim_{x\to 2} \frac{2x^3-11x^2+20x-12}{x^3-x^2-8x+12}$

2	2	−11	20	−12
		4	−14	12
2	2	−7	6	$\boxed{0}$
		4	−6	
2	2	−3	$\boxed{0}$	
		4		
	2	$\boxed{1}$		

2	1	−1	−8	12
		2	2	−12
2	1	1	−6	$\boxed{0}$
		2	6	
2	1	3	$\boxed{0}$	
		2		
	1	$\boxed{5}$		

$$\lim_{x\to 2} \frac{2x^3-11x^2+20x-12}{x^3-x^2-8x+12} = \lim_{x\to 2} \frac{2(x-2)^3+(x-2)^2}{(x-2)^3+5(x-2)^2} = \lim_{x\to 2} \frac{2(x-2)+1}{(x-2)+5} = \frac{1}{5}.$$

   Note that $2(x-2)^3+(x-2)^2 = (2(x-2)+1)(x-2)^2$, $(x-2)^3+5(x-2)^2 = ((x-2)+5)(x-2)^2$ and $(x-2)^2$ is a common factor.

3. Let $f(x) = q(x)(x-2)+3$, where $q(x)$ is a polynomial. Suppose $q(3) = -5$. Explain that there is c $(2 < c < 3)$ such that $f(c) = 0$.

   Solution. Since $f(x)$ is a polynomial, it is continuous. Since $f(x) = q(x)(x-2)+3$, $f(2) = 3$. By assumption, $q(3) = -5$. Hence $f(3) = q(3)(3-2)+3 = -5+3 = -2$. Since 0 is in between 3 and -2, by Proposition 5.3 (Intermediate Value Theorem) there is a number c in the interval $[2,3]$ such that $f(c) = 0$ and $c \neq 2, 3$.

Quiz 6

ID#: **Name:**

1. Find the derivatives of the following functions $y = f(x)$.

   (a) $y = \frac{1}{3}x^3 - \frac{1}{3x^3} + 2\sqrt{x}$, where $x > 0$.

   (b) $y = \frac{x^3 - 1}{x^3 + 1}$

   (c) $y = (x^3 + 1)^9$

   (d) $y = e^{-\frac{1}{2}x^2}$

2. Find the limit of the following. Hint: First compute $(e^{2x})'$.
   $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{4x^2}$

3. Suppose $f(x)$ is differentiable, $g(x) = f'(x) = x^4 - 6x^3 + 12x^2 - 8x$ and $f(1) = 2$.

   (a) Find the equation of the tangent line to the curve $y = f(x)$ at $x = 1$.

   (b) Find the derivative $h(x)$ of $g(x)$, and the derivative of $h(x)$, i.e., find $h(x) = g'(x) = f''(x)$, and $h'(x) = f'''(x)$.

   (c) Determine whether $f(x)$ is increasing, decreasing, a local maximum or a local minimum at $x = 0$. State your reason.

   (d) Determine whether $f(x)$ is increasing, decreasing, a local maximum or a local minimum at $x = 2$. State your reason.

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Solutions to Quiz 6

1. Find the derivatives of the following functions $y = f(x)$.

   (a) $y = \dfrac{1}{3}x^3 - \dfrac{1}{3x^3} + 2\sqrt{x}$, where $x > 0$. $\quad y' = x^2 + \dfrac{1}{x^4} + \dfrac{1}{\sqrt{x}}$.

   $$y' = \left(\frac{1}{3}x^3 - \frac{1}{3}x^{-3} + 2x^{\frac{1}{2}}\right)' = \frac{3x^2}{3} - \frac{-3x^{-4}}{3} + 2\cdot\frac{1}{2}x^{-\frac{1}{2}} = x^2 + x^{-4} + x^{-\frac{1}{2}}. \qquad ((x^a)' = ax^{a-1})$$

   (b) $y = \dfrac{x^3-1}{x^3+1}$. $\quad y' = \dfrac{3x^2(x^3+1) - (x^3-1)3x^2}{(x^3+1)^2} = \dfrac{6x^2}{(x^3+1)^2}$.

   (c) $y = (x^3+1)^9$. $\quad y' = 9(x^3+1)^8(3x^2) = 27x^2(x^3+1)^8$.

   (d) $y = e^{-\frac{1}{2}x^2}$. $\quad y' = e^{-\frac{1}{2}x^2}(-x) = -xe^{-\frac{1}{2}x^2}$.

2. Find the limit of the following.

   $\displaystyle\lim_{x\to 0}\frac{e^{2x} - 1 - 2x}{4x^2}$. $\quad$ Since $\displaystyle\lim_{x\to 0}(e^{2x} - 1 - 2x) = 0 = \lim_{x\to 0}(4x^2)$, $(e^{2x})' = 2e^{2x}$,

   $$\lim_{x\to 0}\frac{e^{2x} - 1 - 2x}{4x^2} = \lim_{x\to 0}\frac{2e^{2x} - 2}{8x} = \lim_{x\to 0}\frac{4e^{2x}}{8} = \frac{1}{2},$$ as $\lim_{x\to 0}(2e^{2x} - 2) = 0 = \lim_{x\to 0}(8x)$.

3. Suppose $f(x)$ is differentiable, $g(x) = f'(x) = x^4 - 6x^3 + 12x^2 - 8x$ and $f(1) = 2$.

   (a) Find the equation of the tangent line to the curve $y = f(x)$ at $x = 1$.
   Solution. Since $f(1) = 2$ and $f'(1) = g(1) = -1$, the equation of the tangent line is:

   $$y = f(1) + f'(1)(x-1) = 2 - (x-1) = -x + 3.$$

   (b) Find the derivative $h(x)$ of $g(x)$, and the derivative of $h(x)$, i.e., find $h(x) = g'(x) = f''(x)$, and $h'(x) = f'''(x)$.
   Solution.

   $$f''(x) = h(x) = 4x^3 - 18x^2 + 24x - 8, \quad f'''(x) = h'(x) = 12x^2 - 36x + 24.$$

   (c) Determine whether $f(x)$ is increasing, decreasing, a local maximum or a local minimum at $x = 0$. State your reason.
   Solution. $f'(0) = 0$ and $f''(0) = g'(0) = h(0) = -8 < 0$. Hence by the second derivative test, $f(x)$ has a local maximum at $x = 0$.

   (d) Determine whether $f(x)$ is increasing, decreasing, a local maximum or a local minimum at $x = 2$. State your reason.
   Solution. By the calculation below using synthetic division, $f'(2) = f''(2) = f'''(2) = 0$ and $f''''(2) = 24\cdot 2 - 36 = 12 > 0$ as $f''''(x) = 24x - 36$. (These values can be found by substitution as well.) Hence by the second derivative test, $f''(x)$ has a local minimum at $x = 2$ with $f''(2) = 0$. Hence $f''(x) > 0$ near $x = 2$ and $f(x)$ has a local minimum at $x = 2$.

   $$\begin{array}{r|rrrrr} 2 & 1 & -6 & 12 & -8 & 0 \\ & & 2 & -8 & 8 & 0 \\ \hline & 1 & -4 & 4 & 0 & \boxed{0} \end{array} \qquad \begin{array}{r|rrrr} 2 & 4 & -18 & 24 & -8 \\ & & 8 & -20 & 8 \\ \hline & 4 & -10 & 4 & \boxed{0} \end{array} \qquad \begin{array}{r|rrr} 2 & 12 & -36 & 24 \\ & & 24 & -24 \\ \hline & 12 & -12 & \boxed{0} \end{array}$$

   Another Solution. $f'(x) = g(x) = x(x-2)^3$. (See the first synthetic division above.) Hence if x is near 2 and $x < 2$, then $f'(x) < 0$, and if $x > 2$, then $f'(x) > 0$. Hence $f(x)$ is decreasing until x reaches 2 and increasing after it. Therefore, $g(x)$ has a local minimum at $x = 2$.

Quiz 7

ID#: **Name:**

1. Find the derivative of $f(x) = x^3 e^{x^3}$.

2. Find $F'(x)$ when $F(x) = \int_0^x t^2(1+t^3)e^{t^3}\,dt$.

3. Evaluate the definite integral $\int_0^1 x^2(1+x^3)e^{x^3}\,dx$.

4. Compute the following.

   (a) $\int (7x^6 - 8x^3 + 9x^2 - 10x + 11)dx$

   (b) $\int \left(3\sqrt{x} + \frac{12}{x^4}\right)dx$

   (c) $\int (e^{2x} + e^{-2x})dx$

   (d) $\int (2x+1)^6 dx$

   (e) $\int_0^3 (t-3)^4 dt$

5. Let $y = y(x)$. Solve the following differential equations.

   (a) $y' = 7x^6 - 8x^3 + 9x^2 - 10x + 11$, $y(1) = 10$.

   (b) $y' = 1 - y$, $y(0) = 0$.

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Solutions to Quiz 7

1. Find the derivative of $f(x) = x^3 e^{x^3}$.

   Solution. By Product Rule and Chain Rule,

$$f'(x) = (x^3 e^{x^3})' = 3x^2 e^{x^3} + x^3 e^{x^3}(3x^2) = 3x^2(1+x^3)e^{x^3}.$$

2. Find $F'(x)$ when $F(x) = \displaystyle\int_0^x t^2(1+t^3)e^{t^3}\,dt$.

   Solution. By Fundamental Theorem of Calculus,

$$F'(x) = x^2(1+x^3)e^{x^3}.$$

3. Evaluate the definite integral $\displaystyle\int_0^1 x^2(1+x^3)e^{x^3}\,dx$.

   Solution. Since $(x^3 e^{x^3})' = 3x^2(1+x^3)e^{x^3}$ by 1,

$$\int_0^1 x^2(1+x^3)e^{x^3}\,dx = \frac{1}{3}\int_0^1 3x^2(1+x^3)e^{x^3}\,dx = \frac{1}{3}\left[x^3 e^{x^3}\right]_0^1 = \frac{e}{3}.$$

4. Compute the following.

   (a) $\displaystyle\int (7x^6 - 8x^3 + 9x^2 - 10x + 11)dx = x^7 - 2x^4 + 3x^3 - 5x^2 + 11x + C.$

   (b) $\displaystyle\int \left(3\sqrt{x} + \frac{12}{x^4}\right)dx = \int \left(3x^{\frac{1}{2}} + 12x^{-4}\right)dx$

$$= \frac{3}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + \frac{12}{-4+1}x^{-4+1} + C = 2x^{\frac{3}{2}} - 4x^{-3} + C = 2(\sqrt{x})^3 - \frac{4}{x^3} + C.$$

   (c) $\displaystyle\int (e^{2x} + e^{-2x})dx = \frac{1}{2}(e^{2x} - e^{-2x}) + C$. By Chain Rule, $(e^{2x})' = 2e^{2x}$, $(e^{-2x})' = -2e^{-2x}$

   (d) $\displaystyle\int (2x+1)^6 dx = \frac{1}{14}\int 14(2x+1)^6 dx = \frac{1}{14}(2x+1)^7 + C.$ $\quad ((2x+1)^7)' = 14(2x+1)^6$

   (e) $\displaystyle\int_0^3 (t-3)^4 dt = \left[\frac{1}{5}(t-3)^5\right]_0^3 = 0 - \frac{1}{5}\cdot(-3)^5 = \frac{243}{5}.$ $\quad (\frac{1}{5}(t-3)^5)' = (t-3)^4$

5. Let $y = y(x)$. Solve the following differential equations.

   (a) $y' = 7x^6 - 8x^3 + 9x^2 - 10x + 11$, $y(1) = 10$. By 4 (a),

   $y = x^7 - 2x^4 + 3x^3 - 5x^2 + 11x + C,\ 10 = y(1) = 1 - 2 + 3 - 5 + 11 + C$. Hence, $C = 2$,

   and $y = x^7 - 2x^4 + 3x^3 - 5x^2 + 11x + 2$.

   (b) $y' = 1 - y$, $y(0) = 0$.

   $y' = \dfrac{1}{1/(1-y)}$ and $h(x) = 1$, $g(y) = 1/(1-y)$ in Proposition 7.4,

$$x + C = -\log|1-y|,\ C = 0,\ \text{as } y(0) = 0 \text{ and } \log 1 = 0,\ 1 - y = e^{-x} \text{ or } y = 1 - e^{-x}.$$

   Note that as $1 - y > 0$ as $x = 0$, we have $\log|1-y| = \log(1-y)$.