6 Inner Product Spaces

6.1 Inner Product, Norm and Distance

Definition 6.1 An *inner product* on a real vector space V is a function that associates a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in V in such a way that the following axioms are satisfied for all vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ in V and all scalars k.

(a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ (Symmetry axiom)

(b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{z} \rangle = \langle \boldsymbol{u}, \boldsymbol{z} \rangle + \langle \boldsymbol{v}, \boldsymbol{z} \rangle$ (Additive axiom)

(c) $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ (Homogeneity axiom)

(d) $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$ (Positivity axiom)
and if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.

A real vector space with an inner product is called a *real inner product space.*

Remarks.

1. Recall that for $V = \boldsymbol{R}^n$, $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$ $(\boldsymbol{u}, \boldsymbol{v} \in V)$ satisfies the conditions above by Theorem 1.2. Hence it is an inner product defined in Definition 6.1.

2. For all vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ in V and all scalars k,

$$\langle \boldsymbol{z}, \boldsymbol{u} + \boldsymbol{v} \rangle = \langle \boldsymbol{z}, \boldsymbol{u} \rangle + \langle \boldsymbol{z}, \boldsymbol{v} \rangle, \ \langle \boldsymbol{u}, k\boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle.$$

3. $\langle \boldsymbol{0}, \boldsymbol{v} \rangle = 0$ for all $\boldsymbol{v} \in V$, as $\langle \boldsymbol{0}, \boldsymbol{v} \rangle = \langle 0\boldsymbol{0}, \boldsymbol{v} \rangle = 0\langle \boldsymbol{0}, \boldsymbol{v} \rangle = 0$.

4. The definition above is only for real vector spaces, and the inequality in (d) is the usual inequality among reals.

5. As for a complex vector space, a similar notion can be defined as in the next definition. Then the following discussion is almost the same.

Definition 6.2 An *inner product* on a complex vector space V is a function that associates a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in V in such a way that the following axioms are satisfied for all vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ in V and all scalars k $(k \in \boldsymbol{C})$.

(a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}$ (Symmetry axiom)

(b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{z} \rangle = \langle \boldsymbol{u}, \boldsymbol{z} \rangle + \langle \boldsymbol{v}, \boldsymbol{z} \rangle$ (Additive axiom)

(c) $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ (Homogeneity axiom)

(d) $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$ (Positively axiom)

and if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$.

A real vector space with an inner product is called a *real inner product space.*

Definition 6.3 If V is an inner product space, then the *norm* (or *length*) of a vector $\boldsymbol{u} \in V$ is denoted by $\|\boldsymbol{u}\|$ and is defined by

$$\|\boldsymbol{u}\| = \langle \boldsymbol{u}, \boldsymbol{u} \rangle^{1/2}.$$

The *distance* between two points (vectors) $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $d(\boldsymbol{u}, \boldsymbol{v})$ and is defined by

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|.$$

Example 6.1 [Excercise 6.1.30] Let A be an invertible $n \times n$ matrix. The following defines an inner product on $\boldsymbol{R}^n$.

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = A\boldsymbol{u} \cdot A\boldsymbol{v} = (A\boldsymbol{u})^T A\boldsymbol{v} = \boldsymbol{u}^T A^T A\boldsymbol{v}.$$

Proof. The properties (a), (b), (c) are obvious. Clearly $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = (A\boldsymbol{u}) \cdot (A\boldsymbol{u}) \geq 0$ by the nonnegativity condition of the dot product in $\boldsymbol{R}^n$. Moreover if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ implies $A\boldsymbol{u} = \mathbf{0}$. Since A is invertible, $\boldsymbol{u} = \mathbf{0}$, and the condition (d) is proved. ∎

Example 6.2 For $A, B \in \mathrm{Mat}_n(\boldsymbol{R})$ let

$$\langle A, B \rangle = \mathrm{tr}(A^T B).$$

Then $\langle A, B \rangle$ defines an inner product on $\mathrm{Mat}_n(\boldsymbol{R})$.

Example 6.3 Let $\boldsymbol{f} = f(x)$ and $\boldsymbol{g} = g(x)$ be two functions on $C[a, b]$, the set of all continuous functions on $[a, b]$. Define

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_a^b f(x)g(x)dx.$$

Then $\langle \boldsymbol{f}, \boldsymbol{g} \rangle$ defines an inner product on $C[a, b]$.

6.2 Properties of Inner Product Space

Theorem 6.1 ((6.2.1) Cauchy-Schwarz Inequality) *If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in a real inner product space, then*

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|.$$

Equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent.

Proof. If $\boldsymbol{u} = \mathbf{0}$, then there is nothing to prove. Assume that $\boldsymbol{u} \neq \mathbf{0}$. Let t be a scalar. Then

$$\|t\boldsymbol{u} + \boldsymbol{v}\|^2 = \langle t\boldsymbol{u} + \boldsymbol{v}, t\boldsymbol{u} + \boldsymbol{v}\rangle = \|\boldsymbol{u}\|^2 t^2 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle t + \|\boldsymbol{v}\|^2.$$

Since the right hand side is a polynomial of degree exactly equal to 2 in t and the left hand side is always nonnegative for all $t \in \boldsymbol{R}$,

$$(\langle \boldsymbol{u}, \boldsymbol{v}\rangle)^2 - \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 \leq 0.$$

Therefore we have the inequality.

Suppose the equality holds. Then there is a real number s such that $\|s\boldsymbol{u}+\boldsymbol{v}\| = 0$. Hence by the property (d) in Definition 6.1, $s\boldsymbol{u} + \boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent. Conversely if $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent, then either $\boldsymbol{u} = \mathbf{0}$ or there exists a real such that $s\boldsymbol{u} + \boldsymbol{v} = \mathbf{0}$. Hence the discriminant above is 0 and we have equality. ∎

If $\boldsymbol{u}$, $\boldsymbol{v}$ are nonzero vectors, then

$$-1 \leq \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \leq 1.$$

Hence there is a unique angle θ such that

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \quad \text{and } 0 \leq \theta \leq \pi.$$

Two vectors $\boldsymbol{u}, \boldsymbol{v} \in V$ are said to be *orthogonal* when $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 0$.

Theorem 6.2 (6.2.2) *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product space V, and k a scalar. Then:*

(a) $\|\boldsymbol{u}\| \geq 0$.

(b) $\|\boldsymbol{u}\| = 0$ *if and only if* $\boldsymbol{u} = \mathbf{0}$.

(c) $\|k\boldsymbol{u}\| = |k|\|\boldsymbol{u}\|$.

(d) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$. *(Triangle inequality)*

Theorem 6.3 (6.2.3) *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product space V, and k a scalar. Then:*

(a) $d(\boldsymbol{u}, \boldsymbol{v}) \geq 0$.

(b) $d(\boldsymbol{u}, \boldsymbol{v}) = 0$ *if and only if* $\boldsymbol{u} = \boldsymbol{v}$.

(c) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$.

(d) $d(\boldsymbol{u}, \boldsymbol{v}) \leq d(\boldsymbol{u}, \boldsymbol{w}) + d(\boldsymbol{w}, \boldsymbol{v})$. *(Triangle inequality)*

Theorem 6.4 ((6.2.4) Generalization Theorem of Pythagoras) *If* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are vectors in an inner vector space, then*

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 \Leftrightarrow \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0.$$

Exercise 6.1 [Quiz 6] Let A be an invertible $m \times n$ matrix. For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^n$ let

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = A\boldsymbol{u} \cdot A\boldsymbol{v} = (A\boldsymbol{u})^T A\boldsymbol{v} = \boldsymbol{u}^T A^T A\boldsymbol{v}.$$

1. Show that $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ satisfies the properties (a), (b) and (c) of an inner product in Definition 6.1.
2. Show that if $\mathcal{N}(A) = \{\boldsymbol{v} \in \boldsymbol{R}^n \mid A\boldsymbol{v} = \boldsymbol{0}\} = \{\boldsymbol{0}\}$, then $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is an inner product.
3. Show that if $m > n$, then $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is not an inner product.
4. Suppose $A^T A$ is invertible. Show that $m \leq n$.