1 Euclidean n-Space

Definition 1.1 If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is called n-space and is denoted by $\boldsymbol{R}^n$.

Definition 1.2 Two vectors $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ in $\boldsymbol{R}^n$ are called *equal* if

$$u_1 = v_2, u_2 = v_2, \ldots, u_n = v_n.$$

The *sum* $\boldsymbol{u} + \boldsymbol{v}$ is defined by

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

and if k is any scalar, the *scalar multiple* $k\boldsymbol{u}$ is defined by

$$k\boldsymbol{u} = (ku_1, ku_2, \ldots, ku_n).$$

Let $\mathbf{0} = (0, 0, \ldots, 0) \in \boldsymbol{R}^n$, $-\boldsymbol{u} = (-u_1, -u_2, \ldots, -u_n)$ and $\boldsymbol{v} - \boldsymbol{u} = \boldsymbol{v} + (-\boldsymbol{u})$ or, interms of components,

$$\boldsymbol{v} - \boldsymbol{u} = (v_1 - u_1, v_2 - u_2, \ldots, v_n - u_n).$$

Theorem 1.1 (4.1.1) *Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ be vectors in $\boldsymbol{R}^n$ and k and m scalars. Then:*

(a) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$. *(Commutativity)*

(b) $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$ *(Associativity)*

(c) $\boldsymbol{u} + \mathbf{0} = \mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$

(d) $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$*; that is* $\boldsymbol{u} - \boldsymbol{u} = \mathbf{0}$.

(e) $k(m\boldsymbol{u}) = (km)\boldsymbol{u}$.

(f) $k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$.

(g) $(k + m)\boldsymbol{u} = k\boldsymbol{u} + m\boldsymbol{u}$.

(h) $1\boldsymbol{u} = \boldsymbol{u}$.

Definition 1.3 Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\boldsymbol{R}^n$. Then *the Euclidean Inner Product* $\boldsymbol{u} \cdot \boldsymbol{v}$ is defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n,$$

the Euclidean norm (or Euclidean lenght) of a vector $\boldsymbol{u}$ is defined by

$$\|\boldsymbol{u}\| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2},$$

and the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

Theorem 1.2 (4.1.2) *Let* $\boldsymbol{u}$, $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *be vectors in* $\boldsymbol{R}^n$ *and* k *a scalar. Then:*

(a) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.

(b) $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$.

(c) $(k\boldsymbol{u}) \cdot \boldsymbol{v} = k(\boldsymbol{u} \cdot \boldsymbol{v})$.

(d) $\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$. *Further,* $\boldsymbol{v} \cdot \boldsymbol{v} = 0$ *if and only if* $\boldsymbol{v} = \boldsymbol{0}$.

Cauchy-Schwarz Inequality in $\boldsymbol{R}^n$.

Theorem 1.3 (4.1.3) *Let* $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ *and* $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ *be vectors in* $\boldsymbol{R}^n$. *Then*

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

Equality holds if and only if $\boldsymbol{v} = k\boldsymbol{u}$ *for some real* k *or* $\boldsymbol{u} = \boldsymbol{0}$.

Theorem 1.4 (4.1.4) *Let* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *be vectors in* $\boldsymbol{R}^n$ *and* k *a scalar. Then:*

(a) $\|u\| \geq 0$.

(b) $\|\boldsymbol{u}\| = 0$ *if and only if* $\boldsymbol{u} = \boldsymbol{0}$.

(c) $\|k\boldsymbol{u}\| = |k|\|\boldsymbol{u}\|$.

(d) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$. *(Triangle inequality)*

Theorem 1.5 (4.1.4) *Let* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *be vectors in* $\boldsymbol{R}^n$ *and* k *a scalar. Then:*

(a) $d(\boldsymbol{u}, \boldsymbol{v}) \geq 0$.

(b) $d(\boldsymbol{u}, \boldsymbol{v}) = 0$ *if and only if* $\boldsymbol{u} = \boldsymbol{v}$.

(c) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$.

(d) $d(\boldsymbol{u}, \boldsymbol{v}) \leq d(\boldsymbol{u}, \boldsymbol{w}) + (\boldsymbol{w}, \boldsymbol{v})$. *(Triangle inequality)*

Exercise 1.1 [Quiz 1] Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be non-zero vectors in $\boldsymbol{R}^n$.

1. Let λ be a real number. Show the following. (Hint: use $\|\boldsymbol{w}\|^2 = \boldsymbol{w} \cdot \boldsymbol{w}$.)

$$\|\lambda\boldsymbol{u} + \boldsymbol{v}\|^2 = \lambda^2\|\boldsymbol{u}\|^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v})\lambda + \|\boldsymbol{v}\|^2.$$

2. Using the fact that $\|\lambda\boldsymbol{u} + \boldsymbol{v}\|^2 \geq 0$ for all real λ and a property of a quadratic function, show the Cauchy-Schwarz Inequality. (Hint: Discriminant (*Hanbetsu-shiki*))

3. Show the equivalence of the following:

$$|\boldsymbol{u} \cdot \boldsymbol{v}| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \Leftrightarrow \text{There exists } \alpha \in \boldsymbol{R} \text{ such that } \boldsymbol{u} = \alpha\boldsymbol{v}.$$