September 6, 2018

MTH103 Linear Algebra I（線形代数学 I）2018

担当教員（Instructors）： このコースは講義（木曜日第 1,2 時限目 N220）は鈴木が担当し、演習（N231, N232）は月曜日第 6・7 時限目は主としてと久能裕一さんと、山本桃果さん、水曜日第 6・7 時限目は久能裕一さんが担当します。さらに、大学院生の神谷葉月さん、黒田絢子さん、景山友樹さんが、サポートについて個々の質問に答えます。コース全体の責任は鈴木が持ちます。コース運営に問題や疑問・要望があれば、早めに鈴木に申し出て下さい。鈴木の研究室 (S309：ドアにスケジュールを貼っておきます) に質問に来て下さってもよいですし、月・水の 13:20-16:20 は S302 Math Help Desk で質問を受け付けています。

講義予定（Tentative Lecture Schedule）

Date	**Title**	**Reading**	**Note**
September 6	1. System of Linear Equations	1.1, 1.2	
September 13	2. Solution Sets of Linear Systems	1.3, 1.4, 1.5	HW1 (1.6)
September 20	3. Linear Transformations	1.7, 1.8, 1.9	HW2 (1.10 & Suppl.)
September 27	4. Matrix Algebra	2.1, 2.2, 2.3	
October 4	5. Properties of Matrix Algebra	2.4, 2.8, 2.9	HW3 (Suppl.)
October 11	6. Determinants	3.1, 3.2	
October 18	7. Applications of Determinants	3.3	
October 25	8. Vectors and Vector Spaces	Handout	
November 1	9. Eigenvalues and Eigenvectors	5.1, 5.2, 5.3	HW4 (Review)
November 8	10. Review		Old Final

このコースの目的（Course Objectives）

線形代数学の基礎を学ぶ。平面，空間のベクトル，内積，外積，行列，階数，行列式，連立 1 次方程式の解法，行列の固有値，固有ベクトル，および平面，空間上の線形写像の行列表示などを取り扱う。

教科書 (Textbook)： Linear Algebra and Its Applications, 5th Edition, by David C. Lay, paper back, Pearson, Global Edition (三省堂で販売しています)

重要（Important Remarks）： すべての講義と演習に時間通りに出席すること。講義や演習の時間を充実したものとすることは、担当教員の責任です。毎週 5 時間程度はクラス外で予習復習の時間を確保してください。教科書の対応する箇所を授業前にざっと読み、授業後に、教科書をしっかりと理解しながらもう一度読み、各節の Practice Problems を解き、最低でも Core Problems として指定してある問題を解いて演習に臨んで下さい。Core Problems は殆ど奇数番の問題で巻末に答えがあります。演習では最初に、Core Problems の解答を黒板に書き説明をしてもらい、それ以外の問題もできるだけ多く解いてもらいます。病気などでやむを得ず講義・演習を欠席した場合には、IT サポート・ツール（Moodle、ホームページ) を利用したり、直接質問に来るなどして早めに欠席した部分を補って下さい。教科書が英語であることは大変だと感じる学生も多いと思いますが、特に自然科学では、早く英語の教科書に慣れることも大切です。

成績評価（Grading Policy）

Homework（今の計画では 4 回）合計点を 100 点満点に換算したもの、演習の時間における、黒板に解答を書いての説明、小テスト（計画では 2 回）の点を合計した演習での点 100 点満点に換算（評価の具体的基準は、授業の運営が落ち着いてから示します）、Final Exam（期末試験期間中に実施）を 200 点満点に換算し、合計で 400 点を満点として評価します。

学修支援（Learning Support）

1. Moodle 3: https://moodle3.icu.ac.jp/course/view.php?id=64 （Enrollment Key: LAI2018）
2. 2012 年度のこの授業は、ICU OCW として公開しています。http://ocw.icu.ac.jp/majors/mth103_2012a/
3. このコースのホームページ：https://icu-hsuzuki.github.io/science/class/linear1/index-j.html
4. Suzuki's Office Hour: Every Wednesday and Friday from 8:50 a.m. - 10:00 a.m. or with appointment. Office: Science Hall S309, Email: *hsuzuki@icu.ac.jp*

Section Summary

1.1 System of Linear Equations; consistent, unique solution, inconsistent

1.2 Row Reduction and Echelon Forms; pivot position, pivot column, reduced row echelon form

1.3 Vector Equations; linear combination, span

1.4 The Matrix Equation $Ax = b$; existence of solutions

1.5 Solution Sets of Linear System; homogeneous linear systems, parametric vector form

1.6 Applications of Linear Systems; Economics Ex.1-4, Chemistry Ex.5-10, Network Ex.11-14

1.7 Linear Independence

1.8 Introduction to Linear Transformations

1.9 The Matrix of a Linear Transformation; standard matrix, reflections, contractions and expansions, shears, projections

1.10 Business Models Ex.1-4, Electrical Networks Ex.5-8, Difference Equations Ex.9-11

2.1 Matrix Operations

2.2 Inverse of a Matrix; algorithm

2.3 Characterization of Invertible Matrices

2.4 Partitioned Matrices

2.5 Matrix Factorizations

2.6 The Leontief Input-Output Model

2.7 Applications to Computer Graphics

2.8 Subspaces of $\mathbb{R}^n$

2.9 Dimension and Rank

3.1 Introduction to Determinants

3.2 Properties of Determinants

3.3 Cramer's Rule, Volume, and Linear Transformations

5.1 Eigenvalues and Eigenvectors; linearly independence of eigenvectors

5.2 The Characteristic Equation

5.3 Diagonalization

Class Schedule 2018

M	W	Th
		9/6 Lecture 1 System of Lin. Eqns. 1.1, 1.2
9/10 Recitation 1 1.1, 1.2	9/12 Recitation 1 1.1, 1.2	9/13 Lecture 2 Soln Sets of Linear Systems 1.3, 1.4, 1.5, HW1
9/17 Recitation 2 1.3, 1.4, 1.5	9/19 Recitation 2 1.3, 1.4, 1.5	9/20 Lecture 3 Linear Transformations 1.7, 1.8, 1.9, HW2
9/24 Recitaiton 3 1.7, 1.8, 1.9	9/26 Recitation 3 1.7, 1.8, 1.9	9/27 Lecture 4 Matrix Algebra 2.1, 2.2
10/1 Recitation 4 2.1, 2.2	10/3 Recitation 4 2.1, 2.2	10/4 Lecture 5 Properties of Matrix Algebra 2.3, 2.4, HW3
10/8 Recitation 5 2.3, 2.4	10/10 Recitation 5 2.3, 2.4	10/11 Lecture 6 Determinants 3.1, 3.2
10/15 Recitation 6 3.1, 3.2	10/17 Recitation 6 3.1, 3.2	10/18 Lecture 7 Applications of Determinants 3.3
10/22 Recitation 7 3.3, Suppl.	10/24 Recitation 7 3.3, Suppl.	10/25 Lecture 8 Vectors and Vector Spaces
10/29 Recitation 8 Handout	10/31 Recitation 8 Handout	11/1 Lecture 9 Eigenvalues and Eigenvectors 5.1, 5.2, 5.3, HW4
11/5 Recitation 9 5.1, 5.2, 5.3	11/7 Recitation 9 5.1, 5.2, 5.3	11/8 Lecture 10 Review
11/12 Recitation 10 Remaining Problems Review		

Instructors

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MA	TA 山本桃果	Yamamoto, Momoko	w/ Kuroda at N231
MB	TA 久能裕一	Kuno, Yuichi	w/ Kamitani at N232
W	**TA 久能裕一**	Kuno, Yuichi	w/ Kageyama at N232

Math Helpdesk: Yamamoto (M) , Kuno (W), from 1:20 p.m. – 4:20 p.m. at SH S302

2018.9.6

Core Problems and Homeworks

schedule (tentative)

	Core Problems	Recitation Problems	M	W
1.1	9, 15, 21, 29		9/10	9/12
1.2	3, 9, 15, 17		9/10	9/12
1.3	5, 11, 17		9/17	9/19
1.4	5, 11, 21		9/17	9/19
1.5	3, 5, 17		9/17	9/19
1.6	HW1			
1.7	5, 11, 17		9/24	9/26
1.8	3, 9, 17		9/24	9/26
1.9	1, 9, 15, 17, 35		9/24	9/26
1.10	HW2			
Suppl.	HW2			
2.1	5, 11, 17, 27		10/1	10/3
2.2	5, 9, 15, 35		10/1	10/3
2.3	3, 13, 21, 27, 33		10/8	10/10
2.4	9, 17, 21		10/8	10/10
2.5	–			
2.6	–			
2.7	–			
2.8	–			
2.9	–			
Suppl.	HW3			
3.1	3, 9, 15, 23		10/15	10/17
3.2	5, 9, 11, 21		10/15	10/17
3.3	5, 11, 30, 32		10/22	10/24
Suppl.	4, 9, 17		10/22	10/24
Vectors	1(a)–(d), 3(a)(b)		10/29	10/31
5.1	3, 5, 29		11/5	11/7
5.2	3, 9, 15		11/5	11/7
5.3	3, 5, 11		11/5	11/7

HW1	Due Sept. 20	Read 1.6 and solve 2 each from Ex. 2–4, Ex. 6–10, Ex. 12–14
HW2	Due Sept. 27	Read 1.10 and solve 1 problem each from Ex. 1–4, 5–8, 9–14 in Section 1.1 and 3 problems from Chap.1 Suppl. Ex. 3, 7, 13, 16, 17, total of 6
HW3	Due Oct.11	6 problems from Chap.2 Suppl. Ex. 3, 4, 5, 11, 13, 14, 16, 18
HW4	Due Nov. 9	Review Problems (to be distributed)

For problems with [M], you may use Linear Algebra Toolkit, SageMath, or Octave and give inputs.

Submit all your homework in the Report Box No.11 in front of Science Hall N201 by 9:00 p.m.

Attend either Monday or Wednesday Recitation Class by your choice.

Solve all core problems of corresponding sections bofore you come to recitation session.

2018.9.6

1 System of Linear Equations

Matrices

Definition 1.1 A *matrix* (or an $m \times n$ *matrix*（行列）) is an $m \times n$ rectangular array of numbers. A matrix with only one column is called a *column vector* or simply a *vector* in $\mathbb{R}^m$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Linear Systems and Augmented Matrices

Definition 1.2 A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a *system of linear equations*（連立一次（線形）方程式） or a *linear system.*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases}$$

where $x_1, x_2, \ldots, x_n$ are the unknowns（未知数）.

A *solution*（解） of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted（代入する）for $x_1, x_2, \ldots, x_n$, respectively. The set of all solutions of the system is called its *solution set* or the *general solution*（一般解）of the system. Two linear systems are *equivalent*（同値）if they have the same solution set.

A system of equations that has no solutions is said to be *inconsistent*（解なし・不能）; if there is at least one solution of the system, it is called *consistent*（解が存在する）.

The *augmented matrix*（拡大係数行列）A or extended coefficient matrix, and the *coefficient matrix*（係数行列）C of this system are defined as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}, \; C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Fundamental Questions About a Linear System

1. Is the system consistent? Does at least one solution exist?
2. If a solution exists, is it the only one? Is the solution unique?
3. If more than one solution exist, what can we say about the solution set?

Elementary operations on equations:

1. (Replacement) Replace one equation by the sum of itself and a multiple of another.
2. (Interchange) Interchange two equations.
3. (Scaling) Multiply an equation through by a nonzero constant.

Elementary row operations:

1. (Replacement) Replace one row by the sum of itself and a multiple of another.
   $[i, j; c]$: Replace row i by the sum of row i and c times row j.
   Add c times row j to row i.
2. (Interchange) Interchange two rows.
   $[i, j]$: Interchange row i and row j.
3. (Scaling) Multiply all entries in a row by a nonzero constant.
   $[i; c]$: Multiply all entries in row i by a nonzero constant c.

Note. The notation above ($[i, j; c]$, $[i, j]$, $[i; c]$) is not in the textbook.

Two matrices are called *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other. (page 22)

Proposition 1.1 (page 23) *If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.*

A *leading entry* of a row refers to the leftmost nonzero entry (in a nonzero row).

Definition 1.3 An $m \times n$ matrix is in *echelon form* (or *row echelon form*（階段行列）) if it has the following properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

It is in *reduced row-echelon form*（既約ガウス行列）if it is in echelon form and it has the following properties:

4. The leading entry in each nonzero row is 1. We call this a *leading* 1.
5. Each leading 1 is the only nonzero entry in the column.

Theorem 1.2 (Theorem 1. (Gauss-Jordan Elimination, p.29)) *Every matrix is row equivalent to one and only one reduced row echelon matrix. (i.e., it can be transformed into a reduced row-echelon form by applying elementary row operations successively finitely many times, and the reduced row-echelon form is uniquely determined).*

Algorithm to Solve a Linear System

Step 1. Write the augmented matrix[1] of the system[2].

$$\begin{cases} x_1 + 0x_2 + x_3 + 0x_4 + x_5 + 3x_6 & = & -1 \\ -x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 - 4x_6 & = & -1 \\ 0x_1 + x_2 - 2x_3 + 3x_4 + 0x_5 - x_6 & = & 3 \\ -2x_1 - 2x_2 + 2x_3 - 6x_4 - 2x_5 - 4x_6 & = & -4\ [resp. 4] \end{cases} \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4\ [4] \end{bmatrix}$$

A system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form $[0, \cdots, 0, b]$ *with* b *nonzero. If a linear system is consistent, then the solution set contains either (i) a unique solution when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.* [3]

Step 2. Use the row reduction algorithm[4] to obtain an equivalent augmented matrix in echelon form[5] by elementary operations ($[i, j; c], [i, j], [i; c]$)[6]. Decide whether the system is consistent. If there is no solution, stop; otherwise go to the next step.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4\ [4] \end{bmatrix} \xrightarrow{[4,1;2]} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & -2 & 4 & -6 & 0 & 2 & -6\ [2] \end{bmatrix} \xrightarrow{[2,1;1]}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & -2 & 4 & -6 & 0 & 2 & -6\ [2] \end{bmatrix} \xrightarrow{[2,3]} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & -2 & 4 & -6 & 0 & 2 & -6\ [2] \end{bmatrix} \xrightarrow{[4,2;2]}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\ [8] \end{bmatrix} \xrightarrow{[1,3;-1]} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 4 & 1\ [0] \\ 0 & 1 & -2 & 3 & 0 & -1 & 3\ [0] \\ 0 & 0 & 0 & 0 & 1 & -1 & -2\ [0] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\ [1] \end{bmatrix}$$

echelon form *reduced echelon form*

The system in [ ] is inconsistent as one of the rows of its echelon matrix is [0 0 0 0 0 0 8].

Each matrix is row equivalent to one and only one reduced row echelon matrix.[7]

Step 3. Continue row reduction to obtain the reduced row echelon form.

Step 4. Write the system of equations corresponding to the matrix obtained in Step 3.

$$\begin{cases} x_1 \quad +x_3 \quad +4x_6 & = & 1 \\ x_2 \quad -2x_3 \quad +3x_4 \quad -x_6 & = & 3 \\ x_5 \quad -x_6 & = & -2 \\ 0 & = & 0 \end{cases} \Longrightarrow \begin{cases} x_1 & = & 1 - x_3 - 4x_6, \\ x_2 & = & 3 + 2x_3 - 3x_4 + x_6, \\ x_3 & & \text{is free}, \\ x_4 & & \text{is free}, \\ x_5 & = & -2 + x_6, \\ x_6 & & \text{is free}. \end{cases}$$

[1]page 20
[2]page 18
[3]Theorem 2 in page 37
[4]page 31–33
[5]page 29
[6]page 22, and class note
[7]Theorem 1 in page 29

Examples

$$\left\{\begin{array}{rcrcr} x & - & 3y & = & 2 \\ x & + & 2y & = & 12 \end{array}\right. \qquad \left[\begin{array}{rrr} 1 & -3 & 2 \\ 1 & 2 & 12 \end{array}\right]$$

$$\left\{\begin{array}{rcrcrcr} 3x & + & y & + & 2z & = & 4 \\ x & + & y & + & z & = & 1 \\ 11x & - & y & + & 5z & = & 17 \end{array}\right. \qquad \left[\begin{array}{rrrr} 3 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \\ 11 & -1 & 5 & 17 \end{array}\right]$$

$$\left[\begin{array}{rrr} 1 & 0 & 8 \\ 0 & 1 & 2 \end{array}\right], \quad \left[\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{rrrr} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$