Review: Solution Sets of Linear System

Example 2.2

$$\begin{cases} x_1 + 0x_2 + x_3 + 0x_4 + x_5 + 3x_6 &= -1 \\ -x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 - 4x_6 &= -1 \\ 0x_1 + x_2 - 2x_3 + 3x_4 + 0x_5 - x_6 &= 3 \\ -2x_1 - 2x_2 + 2x_3 - 6x_4 - 2x_5 - 4x_6 &= -4 \end{cases} \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4 \end{bmatrix}$$

$$\to\to\to \underset{\textit{echelon form}}{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \longrightarrow \underset{\textit{reduced echelon form}}{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\begin{cases} x_1 \quad +x_3 \quad +4x_6 = 1 \\ x_2 \quad -2x_3 \quad +3x_4 \quad -x_6 = 3 \\ x_5 \quad -x_6 = -2 \\ 0 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = 1 - x_3 - 4x_6, \\ x_2 = 3 + 2x_3 - 3x_4 + x_6, \\ x_3 \text{ is free,} \\ x_4 \text{ is free,} \\ x_5 = -2 + x_6, \\ x_6 \text{ is free.} \end{cases}$$

Vector Equation, Matrix Equation

$$A\boldsymbol{x} = x_1\begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 3 \\ -6 \end{bmatrix} + x_5\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} + x_6\begin{bmatrix} 3 \\ -4 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -4 \end{bmatrix}, \text{ where}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 \\ -1 & 0 & -1 & 0 & 0 & -4 \\ 0 & 1 & -2 & 3 & 0 & -1 \\ -2 & -2 & 2 & -6 & -2 & -4 \end{bmatrix}. \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -4 \end{bmatrix}. \text{ We write } A\boldsymbol{x} = \boldsymbol{b}.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ -2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \quad \boldsymbol{p} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \ \boldsymbol{v} = s\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

$s = x_3$, $t = x_4$ and $u = x_6$ are free parameters.

$$\{\boldsymbol{v} \mid A\boldsymbol{v} = \boldsymbol{0}\}$$

$$= \left\{ s\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \middle| \, s, t, u \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

4 Linear Transformations

4.1 Linear independence

Definition 4.1 [page 73] An indexed set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly independent*（一次独立／線形独立） if the vector equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_p\boldsymbol{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ is said to be *linearly dependent*（一次従属／線形従属） if there exist weights $c_1, c_2, \ldots, c_p$ not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p = \mathbf{0}.$$

(This is called a *linear dependence relation* among $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p$

The columns of a matrix A are linearly independent if and only if the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

Example 4.1

$$A\boldsymbol{x} = x_1\begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 3 \\ -6 \end{bmatrix} + x_5\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} + x_6\begin{bmatrix} 3 \\ -4 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ where}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 \\ -1 & 0 & -1 & 0 & 0 & -4 \\ 0 & 1 & -2 & 3 & 0 & -1 \\ -2 & -2 & 2 & -6 & -2 & -4 \end{bmatrix} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6],\ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \text{ and } \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{aligned}
&\{\boldsymbol{v} \mid A\boldsymbol{x} = \mathbf{0}\} \\
&= \left\{ s\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \,\middle|\, s, t, u \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \\
&= \{s\boldsymbol{v}_1 + t\boldsymbol{v}_2 + u\boldsymbol{v}_3 \mid s, t, u \in \mathbb{R}\},\ \text{where } \boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{v}_3 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.
\end{aligned}$$

Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent and $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\}$ is linearly dependent.

Recall that the homogeneous equation $A\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable. (Proposition 2.5 (page 60))

Linear Independence: Let $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n]$ be an $m \times n$ matrix. Then the following are equivalent.

(i) The columns of A, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$, are linearly independent.

(ii) $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = \mathbf{0}$ has only the trivial solution.

(iii) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

(iv) The equation has no free variable.

(v) A has a pivot position in every column.

Example 4.2 (i) If a set $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ contains the zero vector, then the set is linearly dependent.

(ii) If $\boldsymbol{v}_2 = c\boldsymbol{v}_1$, then $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ is linearly dependent.

(iii) A set of two vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Theorem 4.1 (Theorem 7 in page 75) *An indexed set $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vector in S is a linear combination of the others. (In fact, if S is linearly dependent and $\boldsymbol{v}_1 \neq \mathbf{0}$, then some $\boldsymbol{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{j-1}$,)*

Theorem 4.2 (Theorem 8 in page 76) *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.*

4.2 Introduction to Linear Transformations

Let A be an $m \times n$ matrix, $\boldsymbol{x} \in \mathbb{R}^n$. Then $A\boldsymbol{x} \in \mathbb{R}^m$. Hence, each vector $\boldsymbol{x}$ in $\mathbb{R}^n$ corresponds to a vector $\boldsymbol{y}$ in $\mathbb{R}^m$. We write the correspondence T as follows.

$$T : \mathbb{R}^n \to \mathbb{R}^m \ (\boldsymbol{x} \mapsto A\boldsymbol{x}).$$

Definition 4.2 A *transformation*（変換） (or *function*（関数） or *mapping*（写像）) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\boldsymbol{x} \in \mathbb{R}^n$ a vector $T(\boldsymbol{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the *domain*（定義域） of T and $\mathbb{R}^m$ is called the *codomain*（余域） of T. For $\boldsymbol{x} \in \mathbb{R}^n$, the vector $T(\boldsymbol{x})$ is called the *image*（像） of x (under the action of T). The set of all images $T(\boldsymbol{x})$ is called the *range*（値域） of T.

Definition 4.3 [page 82] A transformation (or mapping) T is *linear*（線形[8]） if

(i) $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v}$ in the domain of T.

[8]線形変換／一次変換と言います

(ii) $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$ for all scalars c and all $\boldsymbol{u}$ in the domain of T.

Example 4.3 [Matrix Linear Transformation] If A is an $m \times n$ matrix, the transformation

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \ (\boldsymbol{x} \mapsto A\boldsymbol{x})$$

is a linear transformation.

Let T be a linear transformation. Then

- $T(\mathbf{0}) = \mathbf{0}$.
- $T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$.
- $T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p) = c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \cdots + c_pT(\boldsymbol{v}_p)$.

4.3 Matrix of Linear Transformation

Theorem 4.3 (Theorem 10 in page 88) *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then there exists a unique matrix A such that*

$$T(\boldsymbol{x}) = A\boldsymbol{x} \text{ for all } \boldsymbol{x} \in \mathbb{R}^n.$$

In fact

$$A = [T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \ldots, T(\boldsymbol{e}_n)].$$

Definition 4.4 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping.

1. T is *onto*（上への写像・全射） if each $\boldsymbol{b} \in \mathbb{R}^m$ is the image of at least one $\boldsymbol{x} \in \mathbb{R}^n$.
2. T is *one-to-one*（1 対 1 写像・単射） if each $\boldsymbol{b}$ in $\mathbb{R}^m$ is the image of at most one $\boldsymbol{x} \in \mathbb{R}^m$.

Theorem 4.4 (Theorem 11 and Theorem 12 in pages 93, 94) *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and let A be the standard matrix for T.*

(i) *The following are equivalent.*

   (a) *T is one-to-one.*
   (b) *The equation $T(\boldsymbol{x}) = \mathbf{0}$ has only the trivial solution.*
   (c) *$A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.*
   (d) *The columns of A are linearly independent.*
   (e) *A has a pivot position in every column.*

(ii) *The following are equivalent.*

   (a) *T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.*
   (b) *The columns of A span $\mathbb{R}^m$, i.e., each vector in $\mathbb{R}^m$ is a linear combination of columns of A.*
   (c) *A has a pivot position in every row.*

Geometric Linear Transformation of $\mathbb{R}^2$: Rotations, Reflections, Contractions and Expansions, Shears, Projections