Linear Algebra I Review

Final 2002-2: Consider a system of linear equations with augmented matrix C.

$$C = \begin{bmatrix} 1 & 0 & 6 & 0 & 0 & 3 & 7 \\ 0 & 1 & 2 & -3 & 0 & 1 & 3 \\ 0 & 2 & 4 & -6 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 6 & 0 & 0 & 3 & 7 \\ 0 & 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}.$$

Let $E(i;c)$, $E(i,j)$, and $E(i,j;c)$ be elementary matrices of size 4 corresponding to elementary operations $[i;c]$, $[i,j]$ and $[i,j;c]$.

1. We obtained the reduced echelon form G after applying a sequence of elementary row operations to the matrix C. Describe each step of a sequence of elementary row operations.
2. Find all solutions of the system of linear equations.
3. Find an invertible matrix P of size 4 such that $G = PC$ and express P as a product of elementary matrices.
4. Show that there is only one P satisfying the previous problem.
5. Find P^{-1} and express P^{-1} as a product of elementary matrices.
6. Find four columns of C that are linearly dependent (resp. independent).

Final 2002-5: Let H be the coefficient matrix of the system of linear equations given below.

$$\begin{cases} x + y + \lambda z = a \\ \lambda x + y + z = b \\ x + \lambda y + z = c \end{cases}, \quad H = \begin{bmatrix} 1 & 1 & \lambda \\ \lambda & 1 & 1 \\ 1 & \lambda & 1 \end{bmatrix}.$$

1. Find the condition for λ and a, b, c that the system of linear equations above has infinitely many solutions.
2. Find the condition of λ to satisfy when H is invertible. Find H^{-1} for such λ.

Quiz 2007-6-1: Consider the equation $A\boldsymbol{x} = \boldsymbol{b}$, where A, $\boldsymbol{x}$, $\boldsymbol{b}$ are as follows.

$$A = \begin{bmatrix} 2 & -2 & -4 & 0 \\ -3 & 5 & 4 & 5 \\ 4 & 2 & -5 & 3 \\ 5 & -7 & -3 & 0 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

1. Evaluate $\det(A)$, and determine whether there is no solution, exactly one solution or infinitely many solutions.
2. By Cramer's rule express $x_3 = \frac{\det(B)}{\det(A)}$ as a fraction of two determinants. Write down the matrix B in the numerator.
3. Evaluate $\det(B)$ in the previous problem and find x_3.

Special Determinant:

$$\begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & c_0 \\ -1 & \lambda & 0 & \cdots & 0 & c_1 \\ 0 & -1 & \lambda & \cdots & 0 & c_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \lambda + c_{n-1} \end{vmatrix} = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1} + \lambda^n.$$

Review Special Determinants Discussed at a Lecture:

$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ & \cdots & \cdots & \cdots & \\ b & b & b & \cdots & a \end{vmatrix} = (a+(n-1)b)(a-b)^{n-1}, \quad \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ & & \cdots & \cdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{i>j}(x_i - x_j).$$

Cf. Quiz 2010-6-1, 2: Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be as follows.

$$\boldsymbol{u} = [5, -2, 1]^\top,\ \boldsymbol{v} = [4, -1, 1]^\top,\ \boldsymbol{w} = [1, -1, 0]^\top.$$

1. Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Find both $\cos\theta$ and $\sin\theta$.
2. Compute $\boldsymbol{u} \times \boldsymbol{v}$ and find a vector of length 1 that is a positive scalar multiple of $\boldsymbol{u} \times \boldsymbol{v}$.
3. Find the volume of the parallelopiped (*heiko-6-mentai*) in 3-space determined by the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.
4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ $(\boldsymbol{x} \mapsto \boldsymbol{u} \times \boldsymbol{x})$. Then T is a linear transformation. Find the standard matrix of T.
5. Let $U : \mathbb{R}^3 \to \mathbb{R}$ $(\boldsymbol{x} \mapsto \boldsymbol{u} \cdot \boldsymbol{x})$. Then U is a linear transformation. Find the standard matrix of U.

Linear Reflection: For $\boldsymbol{u} = [u_1, u_2, u_3, u_4]^\top$ be a nonzero vector in $\mathbb{R}^4$, Let

$$\tau_{\boldsymbol{u}} : \mathbb{R}^4 \to \mathbb{R}^4\ (\boldsymbol{x} \mapsto \boldsymbol{x} - \frac{2\boldsymbol{x} \cdot \boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}).$$

1. Show that $\tau_{\boldsymbol{u}}$ is a linear transformation.
2. Let $\boldsymbol{v} = [1, -1, 0, 0]^\top$. Find the standard matrix of $\tau_{\boldsymbol{v}}$.

Diagonalization: Let A be given below.

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

1. Find the characteristic polynomial and the eigenvalues of A.
2. Find A^{100} and A^{25}.
3. Find an invertible matrix P and its inverse P^{-1} such that $P^{-1}AP$ is a diagonal matrix.

To be discussed on November 9, 2017.

Visit ICU OCW (http://ocw.icu.ac.jp/majors/mth103_2012a/), and check Lecture 10 video (After 56'00"), and Lecture 11 video. There is a link from Moodle.) For references, see 'Linear Algebra and Its Applications, Fifth Edition' by David C. Lay