1 System of Linear Equations

Matrices

Definition 1.1 A *matrix* (or an $m \times n$ *matrix* （行列）) is an $m \times n$ rectangular array of numbers. A matrix with only one column is called a *column vector* or simply a *vector* in $\mathbb{R}^m$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Linear Systems and Augmented Matrices

Definition 1.2 A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a *system of linear equations*（連立一次（線形）方程式） or a *linear system.*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{cases}$$

where $x_1, x_2, \ldots, x_n$ are the unknowns（未知数）.

A *solution*（解） of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted（代入する）for $x_1, x_2, \ldots, x_n$, respectively. The set of all solutions of the system is called its *solution set* or the *general solution*（一般解）of the system. Two linear systems are *equivalent*（同値） if they have the same solution set.

A system of equations that has no solutions is said to be *inconsistent*（解なし・不能）; if there is at least one solution of the system, it is called *consistent*（解が存在する）.

The *augmented matrix*（拡大係数行列）A or extended coefficient matrix, and the *coefficient matrix*（係数行列）C of this system are defined as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}, \; C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Fundamental Questions About a Linear System

1. Is the system consistent? Does at least one solution exist?

2. If a solution exists, is it the only one? Is the solution unique?

3. If more than one solution exist, what can we say about the solution set?

Elementary operations on equations:

1. (Replacement) Replace one equation by the sum of itself and a multiple of another.
2. (Interchange) Interchange two equations.
3. (Scaling) Multiply an equation through by a nonzero constant.

Elementary row operations:

1. (Replacement) Replace one row by the sum of itself and a multiple of another.
   $[i, j; c]$: Replace row i by the sum of row i and c times row j.
   Add c times row j to row i.
2. (Interchange) Interchange two rows.
   $[i, j]$: Interchange row i and row j.
3. (Scaling) Multiply all entries in a row by a nonzero constant.
   $[i; c]$: Multiply all entries in row i by a nonzero constant c.

Note. The notation above ($[i, j; c]$, $[i, j]$, $[i; c]$) is not in the textbook.

Two matrices are called *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other. (page 22)

Proposition 1.1 (page 23) *If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.*

A *leading entry* of a row refers to the leftmost nonzero entry (in a nonzero row).

Definition 1.3 An $m \times n$ matrix is in *echelon form* (or *row echelon form*（階段行列）) if it has the following properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

It is in *reduced row-echelon form*（既約ガウス行列）if it is in echelon form and it has the following properties:

4. The leading entry in each nonzero row is 1. We call this a *leading* 1.
5. Each leading 1 is the only nonzero entry in the column.

Theorem 1.2 (Theorem 1. (Gauss-Jordan Elimination, p.29)) *Every matrix is row equivalent to one and only one reduced row echelon matrix. (i.e., it can be transformed into a reduced row-echelon form by applying elementary row operations successively finitely many times, and the reduced row-echelon form is uniquely determined).*

Algorithm to Solve a Linear System

Step 1. Write the augmented matrix[1] of the system[2].

$$\left\{\begin{array}{rcl} x_1+0x_2+x_3+0x_4+x_5+3x_6 &=& -1 \\ -x_1+0x_2-x_3+0x_4+0x_5-4x_6 &=& -1 \\ 0x_1+x_2-2x_3+3x_4+0x_5-x_6 &=& 3 \\ -2x_1-2x_2+2x_3-6x_4-2x_5-4x_6 &=& -4\ [resp.4] \end{array}\right. \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4\ [4] \end{bmatrix}$$

A system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form $[0, \cdots, 0, b]$ with b nonzero. If a linear system is consistent, then the solution set contains either (i) a unique solution when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable. [3]

Step 2. Use the row reduction algorithm[4] to obtain an equivalent augmented matrix in echelon form[5] by elementary operations $([i,j;c],[i,j],[i;c])$[6]. Decide whether the system is consistent. If there is no solution, stop; otherwise go to the next step.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4\ [4] \end{bmatrix} \xrightarrow{[4,1;2]} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & -2 & 4 & -6 & 0 & 2 & -6\ [2] \end{bmatrix} \xrightarrow{[2,1;1]}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & -2 & 4 & -6 & 0 & 2 & -6\ [2] \end{bmatrix} \xrightarrow{[2,3]} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & -2 & 4 & -6 & 0 & 2 & -6\ [2] \end{bmatrix} \xrightarrow{[4,2;2]}$$

$$\underset{\textit{echelon form}}{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\ [8] \end{bmatrix}} \xrightarrow{[1,3;-1]} \underset{\textit{reduced echelon form}}{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 4 & 1\ [0] \\ 0 & 1 & -2 & 3 & 0 & -1 & 3\ [0] \\ 0 & 0 & 0 & 0 & 1 & -1 & -2\ [0] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\ [1] \end{bmatrix}}$$

The system in [ ] is inconsistent as one of the rows of its echelon matrix is [0 0 0 0 0 0 8].

Each matrix is row equivalent to one and only one reduced row echelon matrix.[7]

Step 3. Continue row reduction to obtain the reduced row echelon form.

Step 4. Write the system of equations corresponding to the matrix obtained in Step 3.

$$\left\{\begin{array}{rrrrrrcr} x_1 & & +x_3 & & & +4x_6 &=& 1 \\ & x_2 & -2x_3 & +3x_4 & & -x_6 &=& 3 \\ & & & & x_5 & -x_6 &=& -2 \\ & & & & & 0 &=& 0 \end{array}\right. \Longrightarrow \left\{\begin{array}{lcc} x_1 &=& 1-x_3-4x_6, \\ x_2 &=& 3+2x_3-3x_4+x_6, \\ x_3 & & \text{is free}, \\ x_4 & & \text{is free}, \\ x_5 &=& -2+x_6, \\ x_6 & & \text{is free}. \end{array}\right.$$

[1]page 20
[2]page 18
[3]Theorem 2 in page 37
[4]page 31–33
[5]page 29
[6]page 22, and class note
[7]Theorem 1 in page 29

Examples

$$\begin{cases} x - 3y = 2 \\ x + 2y = 12 \end{cases} \qquad \begin{bmatrix} 1 & -3 & 2 \\ 1 & 2 & 12 \end{bmatrix}$$

$$\begin{cases} 3x + y + 2z = 4 \\ x + y + z = 1 \\ 11x - y + 5z = 17 \end{cases} \qquad \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \\ 11 & -1 & 5 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$