Solutions to Take-Home Quiz 7 (*October 26, 2007*)

Let A be a 5×5 matrix and B a 4×4 matrix given below.

1. Show that $\det(A) = (x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)\det(B)$.

   Sol.

$$
\begin{aligned}
|A| &= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 & {x_5}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 & {x_5}^3 \\ {x_1}^4 & {x_2}^4 & {x_3}^4 & {x_4}^4 & {x_5}^4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 & x_5 \\ {x_1}^2 - {x_5}^2 & {x_2}^2 - {x_5}^2 & {x_3}^2 - {x_5}^2 & {x_4}^2 - {x_5}^2 & {x_5}^2 \\ {x_1}^3 - {x_5}^3 & {x_2}^3 - {x_5}^3 & {x_3}^3 - {x_5}^3 & {x_4}^3 - {x_5}^3 & {x_5}^3 \\ {x_1}^4 - {x_5}^4 & {x_2}^4 - {x_5}^4 & {x_3}^4 - {x_5}^4 & {x_4}^4 - {x_5}^4 & {x_5}^4 \end{vmatrix} \\
&= (-1)^6 \begin{vmatrix} x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 \\ {x_1}^2 - {x_5}^2 & {x_2}^2 - {x_5}^2 & {x_3}^2 - {x_5}^2 & {x_4}^2 - {x_5}^2 \\ {x_1}^3 - {x_5}^3 & {x_2}^3 - {x_5}^3 & {x_3}^3 - {x_5}^3 & {x_4}^3 - {x_5}^3 \\ {x_1}^4 - {x_5}^4 & {x_2}^4 - {x_5}^4 & {x_3}^4 - {x_5}^4 & {x_4}^4 - {x_5}^4 \end{vmatrix} \quad \text{(Cofactor expansion)} \\
&\overset{[4,3;x_5]}{=} (-1)^6 \begin{vmatrix} x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 \\ {x_1}^2 - {x_5}^2 & {x_2}^2 - {x_5}^2 & {x_3}^2 - {x_5}^2 & {x_4}^2 - {x_5}^2 \\ {x_1}^3 - {x_5}^3 & {x_2}^3 - {x_5}^3 & {x_3}^3 - {x_5}^3 & {x_4}^3 - {x_5}^3 \\ {x_1}^4 - x_5{x_1}^3 & {x_2}^4 - x_5{x_2}^3 & {x_3}^4 - x_5{x_3}^3 & {x_4}^4 - x_5{x_3}^3 \end{vmatrix} \\
&\overset{[3,2;x_5]}{=} (-1)^6 \begin{vmatrix} x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 \\ {x_1}^2 - {x_5}^2 & {x_2}^2 - {x_5}^2 & {x_3}^2 - {x_5}^2 & {x_4}^2 - {x_5}^2 \\ {x_1}^3 - x_5{x_1}^2 & {x_2}^3 - x_5{x_2}^2 & {x_3}^3 - x_5{x_3}^2 & {x_4}^3 - x_5{x_4}^2 \\ {x_1}^4 - x_5{x_1}^3 & {x_2}^4 - x_5{x_2}^3 & {x_3}^4 - x_5{x_3}^3 & {x_4}^4 - x_5{x_3}^3 \end{vmatrix} \\
&\overset{[2,1;x_5]}{=} (-1)^6 \begin{vmatrix} x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 \\ {x_1}^2 - x_5x_1 & {x_2}^2 - x_5x_1 & {x_3}^2 - x_5x_1 & {x_4}^2 - x_5x_1 \\ {x_1}^3 - x_5{x_1}^2 & {x_2}^3 - x_5{x_2}^2 & {x_3}^3 - x_5{x_3}^2 & {x_4}^3 - x_5{x_4}^2 \\ {x_1}^4 - x_5{x_1}^3 & {x_2}^4 - x_5{x_2}^3 & {x_3}^4 - x_5{x_3}^3 & {x_4}^4 - x_5{x_3}^3 \end{vmatrix}
\end{aligned}
$$

   Factor out $x_1 - x_5$ from the first column, and $x_2 - x_5$ from the second ...

$$
\begin{aligned}
&= (-1)^6(x_1 - x_5)(x_2 - x_5)(x_3 - x_5)(x_4 - x_5) \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 \end{vmatrix} \\
&= (x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)|B|
\end{aligned}
$$

2. Find $\det(A)$. (Solution only.)

   Sol. This is called the Vandermonde determinant. Please be careful on the indices. There are various expression of products just as Σ notation for summations.

$$
\begin{aligned}
|A| &= (x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)(x_4 - x_3)(x_4 - x_2)(x_4 - x_1) \\
&\quad (x_3 - x_2)(x_3 - x_1)(x_2 - x_1) \\
&= \prod_{j=2}^{5} \prod_{i=1}^{j-1} (x_j - x_i) = \prod_{1 \le i < j \le 5} (x_j - x_i) \\
&= (-1)^{10} \prod_{i=1}^{4} \prod_{j=i+1}^{5} (x_i - x_j) = \prod_{1 \le i < j \le 5}^{4} (x_i - x_j).
\end{aligned}
$$

For the general case, the Vandermonde determinant has the following value.

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ {x_1}^{n-1} & {x_2}^{n-1} & {x_3}^{n-1} & \cdots & {x_n}^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} (x_j - x_i) = (-1)^{\frac{n(n+1)}{2}} \prod_{1\le i<j\le n} (x_i - x_j).$$

The determinant is nonzero if $x_1, x_2, \ldots, x_n$ are all distinct numbers.

3. Let $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be a polynomial satisfying $f(-3) = 2$, $f(-1) = 5$, $f(2) = -3$, $f(3) = 0$ and $f(7) = 100$. Write down a system of linear equations to find a_0, a_1, a_2, a_3, a_4 and explain why the numbers a_0, a_1, a_2, a_3, a_4 are uniquely determined. Do not solve the equation!

Sol.

$$\begin{cases} a_0 + (-3)a_1 + (-3)^2a_2 + (-3)^3a_3 + (-3)^4a_4 & = & 2 \\ a_0 + (-1)a_1 + (-1)^2a_2 + (-1)^3a_3 + (-1)^4a_4 & = & 5 \\ a_0 + 2a_1 + 2^2a_2 + 2^3a_3 + 2^4a_4 & = & -3 \\ a_0 + 3a_1 + 3^2a_2 + 3^3a_3 + 3^4a_4 & = & 0 \\ a_0 + 7a_1 + 7^2a_2 + 7^3a_3 + 7^4a_4 & = & 100 \end{cases}$$

The coefficient matrix C of this system of linear equation is the transpose of A with $x_1 = -3$, $x_2 = -1$. $x_3 = 2$, $x_4 = 3$ and $x_5 = 7$.

$$C = \begin{vmatrix} 1 & -3 & (-3)^2 & (-3)^3 & (-3)^4 \\ 1 & -1 & (-1)^2 & (-1)^3 & (-1)^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 2^2 & 3^3 & 3^4 \\ 1 & 7 & 7^2 & 7^3 & 7^4 \end{vmatrix}$$

$$\begin{aligned} |C| &= |C^T| \\ &= (7-3)(7-2)(7-(-1))(7-(-3))(3-2)(3-(-1))(3-(-3)) \\ &\quad (2-(-1))(2-(-3))((-1)-(-3)). \end{aligned}$$

Hence the determinant of the coefficient matrix is nonzero. Therefore a_0, a_1, a_2, a_3, a_4 are uniquely determined.

4. Explain why there are infinitely many polynomials $g(x) = a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$ satisfying $g(-3) = 2$, $g(-1) = 5$, $g(2) = -3$ and $g(3) = 0$.

Sol. By the previous problem for each n with $g(7) = m$, a_0, a_1, a_2, a_3, a_4 are uniquely determined. They are different if m is distinct. Hence there are infinitely many polynomials $g(x)$ satisfying the conditions.

Other Solution. We can find a polynomial with the conditions such that $a_4 = 0$. Let $g(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ be a polynomial satisfying $g(-3) = 2$, $g(-1) = 5$, $g(2) = -3$ and $g(3) = 0$. Then

$$h(x) = g(x) + a_4(x-(-3))(x-(-1))(x-2)(x-3) \quad (a_4 \text{ is any number.})$$

also satisfies $h(-3) = 2$, $h(-1) = 5$, $h(2) = -3$ and $h(3) = 0$. Hence there are infinitely many polynomials of degree 4 satisfying the conditions.