Solutions to Take-Home Quiz 5 (*October 12, 2007*)

Let A be the 4×4 matrix given below and B the submatrix that remains after 1st row and 2nd colum are deleted from A.

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 0 & -1 & -3 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Let $M_{i,j}$ be the minor of the (i,j) entry of A above, i.e., the determinant of the submatrix after ith row and jth column are deleted from A. In particular, $M_{1,2} = \det(B)$.

1. Find $\mathrm{adj}(B)$, the adjoint of B. (*Not* $\mathrm{adj}(A)$*!*)

   Sol. Let $m_{i,j}$ the minor of the (i,j) entry of B. Then

   $$m_{1,1} = \det\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = 1(-1) - (-1)1 = 0, \; m_{1,2} = \det\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = -1.$$

   Similarly, $m_{1,3} = -1$, $m_{2,1} = -2$, $m_{2,2} = 2$, $m_{2,3} = 0$, $m_{3,1} = 2$, $m_{3,2} = 1$, $m_{3,3} = 1$. Hence

   $$\mathrm{adj}(B) = \begin{bmatrix} m_{1,1} & -m_{1,2} & m_{1,3} \\ -m_{2,1} & m_{2,2} & -m_{2,3} \\ m_{3,1} & -m_{3,2} & m_{3,3} \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

2. Find $\det(B)$ and determine whether or not the matrix B is invertible.

   Sol.

   $$\det(B) \cdot I = B \cdot \mathrm{adj}(B) = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   Hence $\det(B) = 2 \neq 0$. So B is invertible. Actually $B^{-1} = \frac{1}{2}\mathrm{adj}(B)$.

3. Express $\det(A)$ by the cofactor expansion along the 1st row using minors $M_{i,j}$.

   Sol. Let $C_{i,j}$ be the cofactor of the (i,j) entry of A. Then $C_{i,j} = (-1)^{i+j} M_{i,j}$. Hence

   $$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3} = 3M_{1,1} - 2M_{1,2} + M_{1,3}.$$

4. Express $\det(A)$ by the cofactor expansion along the 2nd column using minors $M_{i,j}$.

   Sol.

   $$\det(A) = a_{1,2}C_{1,2} + a_{2,2}C_{2,2} + a_{3,2}C_{3,2} + a_{4,2}C_{4,2} = -2M_{1,2} + 2M_{3,2}.$$

5. Find $\det(A)$.

   Sol. Use the formula in 4. Since $M_{1,2} = 2$, it suffices to find $M_{3,2}$.

   $$\det\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -3 \\ 1 & -1 & -1 \end{pmatrix} = 3((-1)(-1) - (-1)(-3)) - (1(-1) - (1)(-3)) = -8.$$

   Hence $\det(A) = 2M_{1,2} + 2M_{3,2} = -2 \cdot 2 + 2 \cdot (-8) = -20$.