Linear Algebra I *November 16, 2017*

Solutions to Final Exam 2017

(Total: 100 pts, 50% of the grade)

1. Let $\boldsymbol{u} = [1,0,1]^\top$, $\boldsymbol{v} = [1,-1,2]^\top$, $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$, $\boldsymbol{e}_1 = [1,0,0]^\top$, $\boldsymbol{e}_2 = [0,1,0]^\top$ and $\boldsymbol{e}_3 = [0,0,1]^\top$. (10 pts)

   (a) Find $\boldsymbol{w}$ and the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Show work!
   Solution.

$$\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \left[\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \right]^\top = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

$$\text{Volume} = |(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}| = \|\boldsymbol{w}\|^2 = |1+1+1| = |3| = 3.$$

   (b) Find the standard matrix A of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{u}) = \boldsymbol{e}_1$, $T(\boldsymbol{v}) = \boldsymbol{e}_2$ and $T(\boldsymbol{w}) = \boldsymbol{e}_3$. Show work!
   Solution. Let $B = [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$. Then

$$AB = [A\boldsymbol{u}, A\boldsymbol{v}, A\boldsymbol{w}] = [T(\boldsymbol{u}), T(\boldsymbol{v}), T(\boldsymbol{w})] = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] = I.$$

   By Invertible Matrix Theorem, $A = B^{-1}$.

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{[3,1;-1],[2;-1]} \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{[1,2;-1],[3,2;-1]}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array}\right] \xrightarrow{[3;-1/3],[2,3;-1]} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & -1/3 & -1/3 \end{array}\right]$$

   Hence the standard matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1/3 & -2/3 & 1/3 \\ 1/3 & -1/3 & -1/3 \end{bmatrix}.$$

2. Consider the system of linear equations with augmented matrix $C = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7]$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7$ are the columns of C. Let $A = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6]$ be its coefficient matrix. We obtained a row echelon form G after applying a sequence of elementary row operations to the matrix C. (30 pts)

$$C = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 2 & 0 & -4 & 6 & -3 & -10 & 13 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ -3 & 2 & 8 & -13 & -2 & -13 & -9 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ 0 & 0 & 0 & 0 & -3 & -12 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Describe each step of a sequence of elementary row operations to obtain G from C by $[i,j]$, $[i,j;c]$, $[i;c]$ notation. Show work.
   Solution.

$$C \xrightarrow{[2,1;-2]} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & -3 & -12 & 3 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ -3 & 2 & 8 & -13 & -2 & -13 & -9 \end{bmatrix} \xrightarrow{[4,1;3]} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & -3 & -12 & 3 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ 0 & 2 & 2 & -4 & -2 & -10 & 6 \end{bmatrix}$$

$$\xrightarrow{[4,3;-2]} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & -3 & -12 & 3 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{[2,3]} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ 0 & 0 & 0 & 0 & -3 & -12 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = G.$$

   Hence the sequence of operations above is $[2,1;-2]$, $[4,1;3]$, $[4,3;-2]$, $[2,3]$. Another solution is $[2,1;-2]$, $[4,1;3]$, $[2,3]$, $[4,2;-2]$.

(b) Find an invertible matrix P of size 4 such that $G = PC$ and express P as a product of <u>four</u> elementary matrices. Do not forget writing P. Show work.
Solution. P is the matrix obtained by applying the sequence of row operations $[2, 1; -2]$, $[4, 1; 3]$, $[4, 3; -2]$, $[2, 3]$ to the identity matrix of size 4 in this order. Hence,

$$\begin{aligned} P &= E(2,3)E(4,3;-2)E(4,1;3)E(2,1;-2) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 0 & -2 & 1 \end{bmatrix}. \end{aligned}$$

(c) Find an elementary matrix $E \neq I$ such that $EG = G$ and show that there is another invertible matrix Q different from P such that $G = QC$.
Solution. Since the fourth row is zero, $[4, c]$ or $[i, 4; c]$ $(i = 1, 2, 3)$ does not change G. Hence for example $E = E(4; 2)$ and $E(1, 4; -1)$ are possibilities. If $Q = EP$, then $QC = EPC = EG = G$ and $P \neq EP = Q$, as desired.

(d) Find the reduced row echelon form of the matrix C. Show work.
Solution.

$$C \to\to G \overset{[3;-1/3]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{[2,3;1]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 0 & 1 & 1 & -2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(e) Find all solutions of the system of linear equations.
Solution. Let $x_3 = s$, $x_4 = t$ and $x_6 = r$ be free parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2s - 3t - r + 5 \\ -s + 2t + r + 2 \\ s \\ t \\ -4r - 1 \\ r \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

(f) Explain that the linear transformation defined by $T : \mathbb{R}^6 \to \mathbb{R}^4$ $(\boldsymbol{x} \mapsto A\boldsymbol{x})$, i.e., $T(\boldsymbol{x}) = A\boldsymbol{x}$ is NOT onto.
Solution. If T is onto, its standard matrix A has a pivot position in every row. However, an echelon form of A is the first six columns of G and it does not have a pivot position in every row.

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -2 & 2 & 6 & -4 \\ 3 & -1 & 0 & 2 \\ 2 & 1 & -3 & 4 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 7 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Show work!
Solution.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 0 & 2 & 3 & 1 \\ -2 & 2 & 6 & -4 \\ 3 & -1 & 0 & 2 \\ 2 & 1 & -3 & 4 \end{vmatrix} = 6 \begin{vmatrix} 0 & 2 & 1 & 1 \\ -1 & 1 & 1 & -2 \\ 3 & -1 & 0 & 2 \\ 2 & 1 & -1 & 4 \end{vmatrix} = 6 \begin{vmatrix} 0 & 2 & 1 & 1 \\ -1 & -1 & 0 & -3 \\ 3 & -1 & 0 & 2 \\ 2 & 3 & 0 & 5 \end{vmatrix} \\ &= 6 \begin{vmatrix} -1 & -1 & -3 \\ 3 & -1 & 2 \\ 2 & 3 & 5 \end{vmatrix} = 6 \begin{vmatrix} -1 & -1 & -3 \\ 0 & -4 & -7 \\ 0 & 1 & -1 \end{vmatrix} = 6 \cdot (-1)(4 + 7) = -66. \end{aligned}$$

(b) Express x_2 as a quotient (*bun-su*) of determinants when $A\boldsymbol{x} = \boldsymbol{b}$, and write adj$(A)$, the adjugate of A. Don't evaluate the determinants.

$$x_2 = \frac{\begin{vmatrix} 0 & 2 & 3 & 1 \\ -2 & 0 & 6 & -4 \\ 3 & 1 & 0 & 2 \\ 2 & 7 & -3 & 4 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 3 & 1 \\ -2 & 2 & 6 & -4 \\ 3 & -1 & 0 & 2 \\ 2 & 1 & -3 & 4 \end{vmatrix}} \left(= \frac{-240}{-66} = \frac{40}{11} \right), \quad \left(\text{adj}(A) = \begin{bmatrix} 36 & -33 & -24 & -30 \\ 24 & -33 & 6 & -42 \\ -24 & 11 & -6 & 20 \\ -42 & 33 & 6 & 24 \end{bmatrix} \right)$$

$$\text{adj}(A) = \begin{bmatrix} \begin{vmatrix} 2 & 6 & -4 \\ -1 & 0 & 2 \\ 1 & -3 & 4 \end{vmatrix}, & -\begin{vmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 1 & -3 & 4 \end{vmatrix}, & \begin{vmatrix} 2 & 3 & 1 \\ 2 & 6 & -4 \\ 1 & -3 & 4 \end{vmatrix}, & -\begin{vmatrix} 2 & 3 & 1 \\ 2 & 6 & -4 \\ -1 & 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} -2 & 6 & -4 \\ 3 & 0 & 2 \\ 2 & -3 & 4 \end{vmatrix}, & \begin{vmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 2 & -3 & 4 \end{vmatrix}, & -\begin{vmatrix} 0 & 3 & 1 \\ -2 & 6 & -4 \\ 2 & -3 & 4 \end{vmatrix}, & \begin{vmatrix} 0 & 3 & 1 \\ -2 & 6 & -4 \\ 3 & 0 & 2 \end{vmatrix} \\ \begin{vmatrix} -2 & 2 & -4 \\ 3 & -1 & 2 \\ 2 & 1 & 4 \end{vmatrix}, & -\begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 4 \end{vmatrix}, & \begin{vmatrix} 0 & 2 & 1 \\ -2 & 2 & -4 \\ 2 & 1 & 4 \end{vmatrix}, & -\begin{vmatrix} 0 & 2 & 1 \\ -2 & 2 & -4 \\ 3 & -1 & 2 \end{vmatrix} \\ -\begin{vmatrix} -2 & 2 & 6 \\ 3 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix}, & \begin{vmatrix} 0 & 2 & 3 \\ 3 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix}, & -\begin{vmatrix} 0 & 2 & 3 \\ -2 & 2 & 6 \\ 2 & 1 & -3 \end{vmatrix}, & \begin{vmatrix} 0 & 2 & 3 \\ -2 & 2 & 6 \\ 3 & -1 & 0 \end{vmatrix} \end{bmatrix}.$$

4. Let A be the 4×4 matrix, and B an $m \times n$ matrix such that (40 pts)

$$A = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix} = B^\top B, \quad \text{and let } \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find the determinant of A. Show work!

Solution.

$$\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a+3b & b & b & b \\ a+3b & a & b & b \\ a+3b & b & a & b \\ a+3b & b & b & a \end{vmatrix} = (a+3b) \begin{vmatrix} 1 & b & b & b \\ 1 & a & b & b \\ 1 & b & a & b \\ 1 & b & b & a \end{vmatrix} = (a+3b) \begin{vmatrix} 1 & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix}.$$

Hence $|A| = (a+3b)(a-b)^3$.

(b) Explain that if $B^\top B$ is invertible, then $m \geq n$.

Solution. We prove the contrapositive. Suppose $m < n$. Then every column of B cannot be a pivot column, $B\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution $\boldsymbol{v} \neq \boldsymbol{0}$. Then $B\boldsymbol{v} = \boldsymbol{0}$ and $B^\top B\boldsymbol{v} = \boldsymbol{0}$. Since $B^\top B$ is a square matrix, by Invertible Matrix Theorem, $B^\top B$ cannot be invertible. This proves the assertion. ■

(c) Show that if $m < n$, then $a = b$ or $a = -3b$.

Solution. If $m < n$, then by (b), $A = B^\top B$ is not invertible. Hence $\det(A) = (a+3b)(a-b)^3 = 0$. Thus, $a = -3b$ or $a = b$. ■

(d) Show that $\boldsymbol{b}$ is an eigenvector of A. What is the corresponding eigenvalue?

Solution. Since $\boldsymbol{b}$ is the all one vector, every entry of $A\boldsymbol{b}$ is the row sum of A, which is $a+3b$. Hence, $A\boldsymbol{b} = (a+3b)\boldsymbol{b}$, and $\boldsymbol{b}$ is an eigenvector for an eigenvalue $a+3b$. ■

(e) Find the characteristic polynomial and all eigenvalues of A. Show work!

Solution.

$$|A-xI| = \begin{vmatrix} a-x & b & b & b \\ b & a-x & b & b \\ b & b & a-x & b \\ b & b & b & a-x \end{vmatrix} = (a-x+3b)(a-x-b)^3 = (x-(a+3b))(x-(a-b))^3.$$

Note the determinant above is the determinant of A replacing a by $a-x$. Hence the eigenvalues are $a+3b$ of multiplicity 1 and $a-b$ of multiplicity 3 if $b \neq 0$. If $b=0$, then $A = aI$ and the only eigenvalue of A is a of multiplicity four. ■

(f) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Show work!

Solution. If $b=0$, then $A = aI$ and $P = I$, $D = aI$. So assume that $b \neq 0$.

$$A-(a-b)I = \begin{bmatrix} b & b & b & b \\ b & b & b & b \\ b & b & b & b \\ b & b & b & b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence,

$$P = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} a+3b & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{bmatrix}.$$

If J is the all one matrix, then $A = (a-b)I + bJ$. Since an eigenvector of A corresponding to $a-b$ is characterized by the property that its column sum is zero, which is also an eigenvector of J corresponding to 0, we can take

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}. \quad \text{If } J = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } D' = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence,

$$AP = ((a-b)I + bJ)P = P((a-b)I + bD') = PD.$$

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