Linear Algebra I *November 16, 2017*

Final Exam 2017

(Total: 100 pts, 50% of the grade)

ID#: **Name:**

1. Let $\boldsymbol{u} = [1, 0, 1]^\top$, $\boldsymbol{v} = [1, -1, 2]^\top$, $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$, $\boldsymbol{e}_1 = [1, 0, 0]^\top$, $\boldsymbol{e}_2 = [0, 1, 0]^\top$ and $\boldsymbol{e}_3 = [0, 0, 1]^\top$. (10 pts)

   (a) Find $\boldsymbol{w}$ and the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Show work!

   (b) Find the standard matrix A of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{u}) = \boldsymbol{e}_1$, $T(\boldsymbol{v}) = \boldsymbol{e}_2$ and $T(\boldsymbol{w}) = \boldsymbol{e}_3$. Show work!

Points:

1.(*a*)	(*b*)	2.(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	3.(*a*)∗	(*b*)∗	**Total**
4.(*a*)∗	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)∗			*none*	∗	
								5	10	

メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。

2. Consider the system of linear equations with augmented matrix $C = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7]$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7$ are the columns of C. Let $A = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6]$ be its coefficient matrix. We obtained a row echelon form G after applying a sequence of elementary row operations to the matrix C. (30 pts)

$$C = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 2 & 0 & -4 & 6 & -3 & -10 & 13 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ -3 & 2 & 8 & -13 & -2 & -13 & -9 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 5 \\ 0 & 1 & 1 & -2 & -1 & -5 & 3 \\ 0 & 0 & 0 & 0 & -3 & -12 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Describe each step of a sequence of elementary row operations to obtain G from C by $[i, j], [i, j; c], [i; c]$ notation. Show work.

(b) Find an invertible matrix P of size 4 such that $G = PC$ and express P as a product of <u>four</u> elementary matrices. Do not forget writing P. Show work.

(c) Find an elementary matrix $E \neq I$ such that $EG = G$ and show that there is another invertible matrix Q different from P such that $G = QC$.

(d) Find the reduced row echelon form of the matrix C. Show work.

(e) Find all solutions of the system of linear equations.

(f) Explain that the linear transformation defined by $T : \mathbb{R}^6 \to \mathbb{R}^4$ ($\boldsymbol{x} \mapsto A\boldsymbol{x}$), i.e., $T(\boldsymbol{x}) = A\boldsymbol{x}$ is NOT onto.

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -2 & 2 & 6 & -4 \\ 3 & -1 & 0 & 2 \\ 2 & 1 & -3 & 4 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 7 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Show work!

(b) Express x_2 as a quotient (*bun-su*) of determinants when $A\boldsymbol{x} = \boldsymbol{b}$, and write $\mathrm{adj}(A)$, the adjugate of A. Don't evaluate the determinants.

$x_2 =$, $\mathrm{adj}(A) =$

4. Let A be the 4×4 matrix, and B an $m \times n$ matrix such that (40 pts)

$$A = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix} = B^\top B, \quad \text{and let } \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find the determinant of A. Show work!

(b) Explain that if $B^\top B$ is invertible, then $m \geq n$.

(c) Show that if $m < n$, then $a = b$ or $a = -3b$.

(d) Show that $\boldsymbol{b}$ is an eigenvector of A. What is the corresponding eigenvalue?

(e) Find the characteristic polynomial and all eigenvalues of A. Show work!

(f) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Show work!