Linear Algebra I *November 19, 2015*

Solutions to Final Exam 2015

(Total: 100 pts, 50% of the grade)

1. Let $\boldsymbol{u} = [1,4,9]^T$, $\boldsymbol{v} = [1,8,27]^T$, $\boldsymbol{w} = [1,2,3]^T$, $\boldsymbol{e}_1 = [1,0,0]^T$, $\boldsymbol{e}_2 = [0,1,0]^T$ and $\boldsymbol{e}_3 = [0,0,1]^T$. (10 pts)

   (a) Find $\boldsymbol{u} \times \boldsymbol{v}$ and the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Show work!
   Solution.

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{vmatrix} = \left[\begin{vmatrix} 4 & 9 \\ 8 & 27 \end{vmatrix}, -\begin{vmatrix} 1 & 9 \\ 1 & 27 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 1 & 8 \end{vmatrix} \right]^T = \begin{bmatrix} 36 \\ -18 \\ 4 \end{bmatrix}.$$

$$\text{Volume} = |(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}| = |36 \cdot 1 + (-18) \cdot 2 + 4 \cdot 3| = |12| = 12.$$

   (b) Find the standard matrix A of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) = \boldsymbol{u}$, $T(\boldsymbol{e}_2 + \boldsymbol{e}_3) = \boldsymbol{v}$ and $T(\boldsymbol{e}_3) = \boldsymbol{w}$. Show work!
   Solution.

$$T(\boldsymbol{e}_2) = T(\boldsymbol{e}_2 + \boldsymbol{e}_3) - T(\boldsymbol{e}_3) = \boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 1 \\ 8 \\ 27 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 24 \end{bmatrix}.$$

$$T(\boldsymbol{e}_1) = T(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) - T(\boldsymbol{e}_2 + \boldsymbol{e}_3) = \boldsymbol{u} - \boldsymbol{v} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 8 \\ 27 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -18 \end{bmatrix}.$$

   Hence the standard matrix is

$$A = [T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), T(\boldsymbol{e}_3)] = \begin{bmatrix} 0 & 0 & 1 \\ -4 & 6 & 2 \\ -18 & 24 & 3 \end{bmatrix}.$$

2. Consider the system of linear equations with augmented matrix $C = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7]$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7$ are the columns of C. Let $A = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6]$ be its coefficient matrix. We obtained the reduced row echelon form G after applying a sequence of elementary row operations to the matrix C. (30 pts)

$$C = \begin{bmatrix} -2 & 4 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 & 2 \\ 3 & -6 & 2 & 0 & -3 & 1 & 11 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}.$$

   (a) Describe each step of a sequence of elementary row operations to obtain G from C by $[i,j]$, $[i,j;c]$, $[i;c]$ notation. Show work.
   Solution.

$$C \xrightarrow{[1,3]} \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ -2 & 4 & 0 & 1 & 0 & 0 & 3 \\ 3 & -6 & 2 & 0 & -3 & 1 & 11 \end{bmatrix} \xrightarrow{[3,1;2]} \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 7 \\ 3 & -6 & 2 & 0 & -3 & 1 & 11 \end{bmatrix}$$

$$\xrightarrow{[4,1;-3]} \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 2 & 0 & -6 & 1 & 5 \end{bmatrix} \xrightarrow{[4,2;-2]} \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} = G.$$

   Hence the sequence of operations above is $[1,3]$, $[3,1;2]$, $[4,1;-3]$, $[4,2;-2]$.

(b) Find an invertible matrix P of size 4 such that $G = PC$ and express P as a product of elementary matrices. Do not forget writing P. Show work.

Solution. P is the matrix obtained by applying the sequence of row operations $[1,3]$, $[3,1;2]$, $[4,1;-3]$, $[4,2;-2]$ to the identity matrix of size 4 in this order. Hence

$$\begin{aligned} P &= E(4,2;-2)E(4,1;-3)E(3,1;2)E(1,3) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \end{aligned}$$

(c) Explain (without computation) that $P^{-1} = [\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_6]$.

Solution. Let $Q = [\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_6]$. Since $P\boldsymbol{c}_1, P\boldsymbol{c}_3, P\boldsymbol{c}_4, P\boldsymbol{c}_6$ are the corresponding columns of G, which are $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$,

$$PQ = P[\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_6] = [P\boldsymbol{c}_1, P\boldsymbol{c}_3, P\boldsymbol{c}_4, P\boldsymbol{c}_6] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

So $PQ = I$. Since P is a product of elementary matrices, P is invertible and (or By IMT,) $Q = [\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_6] = P^{-1}$.

(d) Find all solutions of the system of linear equations.

Solution. Let $x_2 = s$ and $x_5 = t$ be free parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2s - t + 2 \\ s \\ 3t \\ -2t + 7 \\ t \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 7 \\ 0 \\ 5 \end{bmatrix} + s \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

(e) Explain that the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbb{R}^4$.

Solution. Since A has pivot position in each row, the last column of the augmented matrix $[A, \boldsymbol{b}]$ cannot be a pivot column. Hence $A\boldsymbol{x} = \boldsymbol{b}$ is always consistent for all $\boldsymbol{b} \in \mathbb{R}^4$.

(f) Explain that the linear transformation defined by $T : \mathbb{R}^6 \to \mathbb{R}^4$ ($\boldsymbol{x} \mapsto A\boldsymbol{x}$), i.e., $T(\boldsymbol{x}) = A\boldsymbol{x}$ is NOT one-to-one.

Solution. Since A is a 4×6 matrix, there is a column which is not a pivot column. Hence if $T(\boldsymbol{x}) = A\boldsymbol{x} = \boldsymbol{b}$ is consistent, there is a free parameter and T is not one-to-one.

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix} 3 & -5 & 7 & 9 \\ 1 & -2 & 3 & -1 \\ -2 & 4 & -5 & -3 \\ 0 & 1 & -2 & 3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Show work!

Solution.

$$\det(A) = \begin{vmatrix} 3 & -5 & 7 & 9 \\ 1 & -2 & 3 & -1 \\ -2 & 4 & -5 & -3 \\ 0 & 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -2 & 12 \\ 1 & -2 & 3 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & 1 & -2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 12 \\ 0 & 1 & -5 \\ 1 & -2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 12 \\ 0 & 1 & -5 \\ 0 & 0 & -9 \end{vmatrix} = 9.$$

(b) Express x_4 as a quotient (*bun-su*) of determinants when $A\boldsymbol{x} = \boldsymbol{b}$, and write $\operatorname{adj}(A)$, the adjugate of A. Don't evaluate the determinants.

$$x_4 = \frac{\begin{vmatrix} 3 & -5 & 7 & 2 \\ 1 & -2 & 3 & 0 \\ -2 & 4 & -5 & 1 \\ 0 & 1 & -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -5 & 7 & 9 \\ 1 & -2 & 3 & -1 \\ -2 & 4 & -5 & -3 \\ 0 & 1 & -2 & 3 \end{vmatrix}} \left(= \frac{3}{9} = \frac{1}{3}\right), \quad \left(\text{adj}(A) = \begin{bmatrix} 0 & 27 & 9 & 18 \\ 7 & 15 & 18 & 2 \\ 5 & 3 & 9 & -5 \\ 1 & -3 & 0 & -1 \end{bmatrix},\right)$$

$$\text{adj}(A) = \begin{bmatrix}
\begin{vmatrix} -2 & 3 & -1 \\ 4 & -5 & -3 \\ 1 & -2 & 3 \end{vmatrix}, & -\begin{vmatrix} -5 & 7 & 9 \\ 4 & -5 & -3 \\ 1 & -2 & 3 \end{vmatrix}, & \begin{vmatrix} -5 & 7 & 9 \\ -2 & 3 & -1 \\ 1 & -2 & 3 \end{vmatrix}, & -\begin{vmatrix} -5 & 7 & 9 \\ -2 & 3 & -1 \\ 4 & -5 & -3 \end{vmatrix} \\
-\begin{vmatrix} 1 & 3 & -1 \\ -2 & -5 & -3 \\ 0 & -2 & 3 \end{vmatrix}, & \begin{vmatrix} 3 & 7 & 9 \\ -2 & -5 & -3 \\ 0 & -2 & 3 \end{vmatrix}, & -\begin{vmatrix} 3 & 7 & 9 \\ 1 & 3 & -1 \\ 0 & -2 & 3 \end{vmatrix}, & \begin{vmatrix} 3 & 7 & 9 \\ 1 & 3 & -1 \\ -2 & -5 & -3 \end{vmatrix} \\
\begin{vmatrix} 1 & -2 & -1 \\ -2 & 4 & -3 \\ 0 & 1 & 3 \end{vmatrix}, & -\begin{vmatrix} 3 & -5 & 9 \\ -2 & 4 & -3 \\ 0 & 1 & 3 \end{vmatrix}, & \begin{vmatrix} 3 & -5 & 9 \\ 1 & 2 & -1 \\ 0 & 1 & 3 \end{vmatrix}, & -\begin{vmatrix} 3 & -5 & 9 \\ 1 & 2 & -1 \\ -2 & 4 & -3 \end{vmatrix} \\
-\begin{vmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 0 & 1 & -2 \end{vmatrix}, & \begin{vmatrix} 3 & -5 & 7 \\ -2 & 4 & -5 \\ 0 & 1 & -2 \end{vmatrix}, & -\begin{vmatrix} 3 & -5 & 7 \\ 1 & -2 & 3 \\ 0 & 1 & -2 \end{vmatrix}, & \begin{vmatrix} 3 & -5 & 7 \\ 1 & -2 & 3 \\ -2 & 4 & -5 \end{vmatrix}
\end{bmatrix}.$$

4. Let A be the 3×3 matrix and B the 4×4 matrix given below, where a, b, c and d are real numbers. (20 pts)

$$A = \begin{bmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}, \quad f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3.$$

(a) Find the determinant of A. Show work!

Solution.

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} = \begin{vmatrix} 1 & b & b^2 \\ 0 & c-b & c^2-b^2 \\ 0 & d-b & d^2-b^2 \end{vmatrix} = \begin{vmatrix} c-b & c^2-b^2 \\ d-b & d^2-b^2 \end{vmatrix} \\ &= \begin{vmatrix} c-b & c^2-cb \\ d-b & d^2-db \end{vmatrix} = (c-b)(d-b)\begin{vmatrix} 1 & c \\ 1 & d \end{vmatrix} = (c-b)(d-b)(d-c). \end{aligned}$$

$[2,1;-1] \to [3,1;-1] \to$ cofactor expansion along the 1st coolmn $\to [2,1;-b]_c \to$ factor out $(c-b)(d-b) \to$ evaluate 2×2 matrix.

(b) Find the determinant of B. Show work!

Solution.

$$\begin{aligned} |B| &= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 & b^3-a^3 \\ c-a & c^2-a^2 & c^3-a^3 \\ d-a & d^2-a^2 & d^3-a^3 \end{vmatrix} \\ &= \begin{vmatrix} b-a & b^2-a^2 & b^3-b^2a \\ c-a & c^2-a^2 & c^3-c^2a \\ d-a & d^2-a^2 & d^3-d^2a \end{vmatrix} = \begin{vmatrix} b-a & b^2-ba & b^3-b^2a \\ c-a & c^2-ca & c^3-c^2a \\ d-a & d^2-da & d^3-d^2a \end{vmatrix} \\ &= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c). \end{aligned}$$

$[2,1;-1] \to [3,1;-1] \to [4,1;-1] \to$ cofactor expansion along the first column $\to [3,2;-a]_c \to$ $[2,1;-a] \to$ factor out $(b-a)(c-a)(d-a) \to$ apply (a).

(c) Suppose a, b, c, d are distinct. Using (b) and show that if $f(a) = f(b) = f(c) = f(d) = 0$, then $f_0 = f_1 = f_2 = f_3 = 0$ and $f(x) = 0$.

Solution.

$$\begin{array}{rcccl} 0 & = & f(a) & = & f_0 + f_1 a + f_2 a^2 + f_3 a^3 \\ 0 & = & f(b) & = & f_0 + f_1 b + f_2 b^2 + f_3 b^3 \\ 0 & = & f(c) & = & f_0 + f_1 c + f_2 c^2 + f_3 c^3 \\ 0 & = & f(d) & = & f_0 + f_1 d + f_2 d^2 + f_3 d^3 \end{array}, \quad \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By (b) the determinant of the coefficient matrix is not zero as a, b, c, d are distinct. Therefore, B is invertible and $f_0 = f_1 = f_2 = f_3 = 0$ and $f(x) = 0$.

5. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 8 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix}$. (20 pts)

(a) Show that 8 is an eigenvalue of A by finding an eigenvector. Show work!

Solution.

$$A - 8I = \begin{bmatrix} -8 & 1 & 0 \\ 8 & -5 & 2 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 2 \\ 8 & -5 & 2 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 8 & -5 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A\boldsymbol{v}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 8 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 16 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 8 \\ 16 \end{bmatrix}, \text{ where an eigenvector } \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 8 \\ 16 \end{bmatrix}.$$

(b) Find the characteristic polynomial and all eigenvalues of A. Show work!

Solution.

$$\begin{aligned} \det(A - xI) &= \begin{vmatrix} -x & 1 & 0 \\ 8 & 3-x & 2 \\ 0 & 4 & 6-x \end{vmatrix} = \begin{vmatrix} 8-x & 8-x & 8-x \\ 8 & 3-x & 2 \\ 0 & 4 & 6-x \end{vmatrix} \\ &= (8-x) \begin{vmatrix} 1 & 1 & 1 \\ 8 & 3-x & 2 \\ 0 & 4 & 6-x \end{vmatrix} = (8-x) \begin{vmatrix} 1 & 1 & 1 \\ 0 & -5-x & -6 \\ 0 & 4 & 6-x \end{vmatrix} \\ &= (8-x)(24 - 30 - x + x^2) = -(x-8)(x-3)(x+2). \end{aligned}$$

Hence the characteristic polynomial is $-(x-8)(x-3)(x+2)$ and $8, 2, -3$ are eigenvalues.

(c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Show work!

Solution.

$$A - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 8 & 0 & 2 \\ 0 & 4 & 3 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} 2 & 1 & 0 \\ 8 & 5 & 2 \\ 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Then $A\boldsymbol{v}_1 = 8\boldsymbol{v}_1, A\boldsymbol{v}_2 = 3\boldsymbol{v}_2, A\boldsymbol{v}_3 = -2\boldsymbol{v}_3$. Therefore,

$$P = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 3 & -2 \\ 16 & -4 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Note that

$$AP = A[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = [A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3] = [8\boldsymbol{v}_1, 3\boldsymbol{v}_2, -2\boldsymbol{v}_3] = PD.$$

Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are eigenvectors corresponding to distinct eigenvectors $8, 3, -2$, they are linearly independent and P is invertible.

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