Linear Algebra I *November 19, 2015*

Final Exam 2015

(Total: 100 pts, 50% of the grade)

ID#: **Name:**

1. Let $\boldsymbol{u} = [1, 4, 9]^T$, $\boldsymbol{v} = [1, 8, 27]^T$, $\boldsymbol{w} = [1, 2, 3]^T$, $\boldsymbol{e}_1 = [1, 0, 0]^T$, $\boldsymbol{e}_2 = [0, 1, 0]^T$ and $\boldsymbol{e}_3 = [0, 0, 1]^T$. (10 pts)

   (a) Find $\boldsymbol{u} \times \boldsymbol{v}$ and the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Show work!

   (b) Find the standard matrix A of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) = \boldsymbol{u}$, $T(\boldsymbol{e}_2 + \boldsymbol{e}_3) = \boldsymbol{v}$ and $T(\boldsymbol{e}_3) = \boldsymbol{w}$. Show work!

Points:

1.(*a*)	(*b*)	2.(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	3.(*a*)*	(*b*)*	**Total**
4.(*a*)	(*b*)*	(*c*)	5.(*a*)	(*b*)	(*c*)*			*none*	*	
								5	10	

メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。

2. Consider the system of linear equations with augmented matrix $C = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7]$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_7$ are the columns of C. Let $A = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6]$ be its coefficient matrix. We obtained the reduced row echelon form G after applying a sequence of elementary row operations to the matrix C. (30 pts)

$$C = \begin{bmatrix} -2 & 4 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 & 2 \\ 3 & -6 & 2 & 0 & -3 & 1 & 11 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}.$$

(a) Describe each step of a sequence of elementary row operations to obtain G from C by $[i, j], [i, j; c], [i; c]$ notation. Show work.

(b) Find an invertible matrix P of size 4 such that $G = PC$ and express P as a product of elementary matrices. Do not forget writing P. Show work.

(c) Explain (without computation) that $P^{-1} = [\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_6]$.

(d) Find all solutions of the system of linear equations.

(e) Explain that the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbb{R}^4$.

(f) Explain that the linear transformation defined by $T : \mathbb{R}^6 \to \mathbb{R}^4$ ($\boldsymbol{x} \mapsto A\boldsymbol{x}$), i.e., $T(\boldsymbol{x}) = A\boldsymbol{x}$ is NOT one-to-one.

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix} 3 & -5 & 7 & 9 \\ 1 & -2 & 3 & -1 \\ -2 & 4 & -5 & -3 \\ 0 & 1 & -2 & 3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Show work!

(b) Express x_4 as a quotient (*bun-su*) of determinants when $A\boldsymbol{x} = \boldsymbol{b}$, and write $\text{adj}(A)$, the adjugate of A. Don't evaluate the determinants.

$x_4 =$, $\text{adj}(A) =$

4. Let A be the 3×3 matrix and B the 4×4 matrix given below, where a, b, c and d are real numbers. (20 pts)

$$A = \begin{bmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}, \quad f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3.$$

(a) Find the determinant of A. Show work!

(b) Find the determinant of B. Show work!

(c) Suppose a, b, c, d are distinct. Using (b), show that if $f(a) = f(b) = f(c) = f(d) = 0$, then $f_0 = f_1 = f_2 = f_3 = 0$ and $f(x) = 0$.

5. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 8 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix}$. (20 pts)

(a) Show that 8 is an eigenvalue of A by finding an eigenvector. Show work!

(b) Find the characteristic polynomial and all eigenvalues of A. Show work!

(c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Show work!