Linear Algebra I *November 21, 2013*

Solutions to Final Exam 2013

(Total: 100 pts, 40% of the grade)

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a transformation defined by: (30 pts)

$$T(x_1, x_2, x_3, x_4) = (3x_1 + x_2 - x_4, x_1 + 2x_2 - 3x_3 + 3x_4, -2x_1 + 4x_2 - 2x_3 + 5x_4).$$

(a) Show that T is a linear transformation.
Solution. Let $\boldsymbol{a} = (a_1, a_2, a_3, a_4)$, $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$, and $c, d \in \mathbb{R}$. We need to show that

$$T(c\boldsymbol{a} + d\boldsymbol{b}) = cT(\boldsymbol{a}) + dT(\boldsymbol{b}).$$

$$\begin{aligned}
&T(c\boldsymbol{a} + d\boldsymbol{b}) \\
&= T(ca_1 + db_1, ca_2 + db_2, ca_3 + db_3, ca_4 + db_4) \\
&= (3(ca_1 + db_1) + (ca_2 + db_2) - (ca_4 + db_4), \\
&\quad (ca_1 + db_1) + 2(ca_2 + db_2) - 3(ca_3 + db_3) + 3(ca_4 + db_4), \\
&\quad -2(ca_1 + db_1) + 4(ca_2 + db_2) - 2(ca_3 + db_3) + 5(ca_4 + db_4)), \\
&= c(3a_1 + a_2 - a_4, a_1 + 2a_2 - 3a_3 + 3a_4, -2a_1 + 4a_2 - 2a_3 + 5a_4) \\
&\quad + d(3b_1 + b_2 - b_4, b_1 + 2b_2 - 3b_3 + 3b_4, -2b_1 + 4b_2 - 2b_3 + 5b_4) \\
&= cT(\boldsymbol{a}) + dT(\boldsymbol{b}).
\end{aligned}$$

(b) Find the standard matrix $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4]$ for the linear transformation T.
Solution. A satisfies the following:

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 - x_4 \\ x_1 + 2x_2 - 3x_3 + 3x_4 \\ -2x_1 + 4x_2 - 2x_3 + 5x_4 \end{bmatrix}. \ \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\boldsymbol{v}_1 = T(1,0,0,0) = T(\boldsymbol{e}_1) = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \boldsymbol{v}_2 = T(\boldsymbol{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \boldsymbol{v}_3 = T(\boldsymbol{e}_3) = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}, \boldsymbol{v}_4 = T(\boldsymbol{e}_4) = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}.$$

Thus

$$A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4] = \begin{bmatrix} 3 & 1 & 0 & -1 \\ 1 & 2 & -3 & 3 \\ -2 & 4 & -2 & 5 \end{bmatrix}.$$

(c) Find $\boldsymbol{v}_1 \times \boldsymbol{v}_2$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are in (b).
Solution. Let $\boldsymbol{e}_1 = [1,0,0]^T$, $\boldsymbol{e}_2 = [0,1,0]^T$, $\boldsymbol{e}_3 = [0,0,1]^T$.

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 3 & 1 & -2 \\ 1 & 2 & 4 \end{vmatrix} = \left[\begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix}, -\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \right]^T = \begin{bmatrix} 8 \\ -14 \\ 5 \end{bmatrix}.$$

(d) Find the volume of the parallelepiped determined by $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, where $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are in (b).
Solution.

$$\begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 4 \\ 0 & -3 & -2 \end{vmatrix} = (\boldsymbol{v}_1 \times \boldsymbol{v}_2) \cdot \boldsymbol{v}_3 = (-14) \cdot (-3) + 5 \cdot (-2) = 32.$$

Hence the volume is $|32| = 32$. ■

(e) Determine whether T is one-to-one. Explain your answer.
Solution. Since a set of four vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ in $\mathbb{R}^3$ is not linearly independent, A is not one-to-one. ■

(f) Determine whether T is onto. Explain your answer.

Solution. By (d) the set of first three columns $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent, so A has pivot positions in all three rows. Hence T is onto. ■

2. Let A be the following 4×4 matrix and a, b, c, d real numbers. (25 pts)

$$A = \begin{bmatrix} 1 & x_1 & {x_1}^2 & {x_1}^3 \\ 1 & x_2 & {x_2}^2 & {x_2}^3 \\ 1 & x_3 & {x_3}^2 & {x_3}^3 \\ 1 & x_4 & {x_4}^2 & {x_4}^3 \end{bmatrix}. \quad f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \text{ is called a } \underline{\text{cubic polynomial}}.$$

(a) Show that $\det(A) = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3)$.

Solution.

$$\begin{aligned}
|A| &= \begin{vmatrix} 1 & x_1 & {x_1}^2 & {x_1}^3 \\ 1 & x_2 & {x_2}^2 & {x_2}^3 \\ 1 & x_3 & {x_3}^2 & {x_3}^3 \\ 1 & x_4 & {x_4}^2 & {x_4}^3 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & {x_1}^2 & {x_1}^3 \\ 0 & x_2 - x_1 & {x_2}^2 - {x_1}^2 & {x_2}^3 - {x_1}^3 \\ 0 & x_3 - x_1 & {x_3}^2 - {x_1}^2 & {x_3}^3 - {x_1}^3 \\ 0 & x_4 - x_1 & {x_4}^2 - {x_1}^2 & {x_4}^3 - {x_1}^3 \end{vmatrix} \\
&= \begin{vmatrix} x_2 - x_1 & {x_2}^2 - {x_1}^2 & {x_2}^3 - {x_1}^3 \\ x_3 - x_1 & {x_3}^2 - {x_1}^2 & {x_3}^3 - {x_1}^3 \\ x_4 - x_1 & {x_4}^2 - {x_1}^2 & {x_4}^3 - {x_1}^3 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & {x_2}^2 - {x_1}^2 & {x_2}^3 - x_1{x_2}^2 \\ x_3 - x_1 & {x_3}^2 - {x_1}^2 & {x_3}^3 - x_1{x_2}^2 \\ x_4 - x_1 & {x_4}^2 - {x_1}^2 & {x_4}^3 - x_1{x_4}^2 \end{vmatrix} \\
&= \begin{vmatrix} x_2 - x_1 & {x_2}^2 - x_1x_2 & {x_2}^3 - x_1{x_2}^2 \\ x_3 - x_1 & {x_3}^2 - x_1x_2 & {x_3}^3 - x_1{x_2}^2 \\ x_4 - x_1 & {x_4}^2 - x_1x_4 & {x_4}^3 - x_1{x_4}^2 \end{vmatrix} \\
&= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \begin{vmatrix} 1 & x_2 & {x_2}^2 \\ 1 & x_2 & {x_2}^2 \\ 1 & x_4 & {x_4}^2 \end{vmatrix} \\
&= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \begin{vmatrix} 1 & x_2 & {x_2}^2 \\ 0 & x_3 - x_2 & {x_3}^2 - {x_2}^2 \\ 0 & x_4 - x_2 & {x_4}^2 - {x_2}^2 \end{vmatrix} \\
&= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \begin{vmatrix} x_3 - x_2 & {x_3}^2 - {x_2}^2 \\ x_4 - x_2 & {x_4}^2 - {x_2}^2 \end{vmatrix} \\
&= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2) \begin{vmatrix} 1 & x_3 + x_2 \\ 1 & x_4 + x_2 \end{vmatrix} \\
&= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3).
\end{aligned}$$

(b) Explain that a cubic polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is uniquely determined when $f(1) = 2, f(2) = 0, f(3) = 1, f(4) = 3$.

Solution. Since

$$\begin{array}{rcccl} f(1) &=& a_0 + a_1 + a_2 + a_3 &=& 2 \\ f(2) &=& a_0 + 2a_1 + 2^2 a_2 + 2^3 a_3 &=& 0 \\ f(3) &=& a_0 + 3a_1 + 3^2 a_2 + 3^3 a_3 &=& 1 \\ f(4) &=& a_0 + 4a_1 + 4^2 a_2 + 4^3 a_3 &=& 3 \end{array}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}$$

is the coefficient matrix. By (a), $\det(B) = (2-1)(3-1)(4-1)(3-2)(4-2)(4-3) \neq 0$. Hence B is invertible and a_0, a_1, a_2, a_3 and $f(x)$ is uniquely determined. ■

(c) Find a_3 in (b) by Cramer's rule. Don't evaluate determinants.

Solution.

$$a_3 = \frac{\begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2^2 & 0 \\ 1 & 3 & 3^2 & 1 \\ 1 & 4 & 4^2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix}}.$$

(d) Suppose x_1, x_2, x_3, x_4 are distinct. Explain that a cubic polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ is uniquely determined when $f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3, f(x_4) = y_4$ for any y_1, y_2, y_3, y_4.

Solution. Since

$$\begin{array}{rcccl} f(1) &=& a_0 + x_1a_1 + {x_1}^2a_2 + {x_3}^3a_3 &=& y_1 \\ f(2) &=& a_0 + x_2a_1 + {x_2}^2a_2 + {x_2}^3a_3 &=& y_2 \\ f(3) &=& a_0 + x_3a_1 + {x_3}^2a_2 + {x_3}^3a_3 &=& y_3 \\ f(4) &=& a_0 + x_4a_1 + {x_4}^2a_2 + {x_4}^3a_3 &=& y_4 \end{array},$$

A is the coefficient matrix. Since x_1, x_2, x_3, x_4 are distinct, by (a),

$$\det(A) = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3) \neq 0.$$

Hence A is invertible and c_0, c_1, c_2, c_3 and $f(x)$ is uniquely determined. ■

3. Let A and B be matrices given below. (25 pts)

$$A = \begin{bmatrix} 3 & -5 & -5 & -4 & -2 \\ -3 & 4 & 2 & 6 & 6 \\ -3 & 3 & 0 & 6 & 9 \\ -3 & 1 & -4 & 7 & 8 \\ -3 & 6 & 6 & 6 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 & -2 & -3 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & -2 & -5 & 2 & 7 \\ 0 & -2 & -4 & 1 & -1 \\ 0 & 3 & 6 & 0 & -2 \end{bmatrix}.$$

(a) The matrix B is obtained from the matrix A by applying a sequence of elementary row operations. Find (i) such a sequence of elementary row operations, (ii) a matrix P such that $PA = B$, and (iii) $\det(P)$.

Solution.

$$\begin{aligned} A &= \begin{bmatrix} 3 & -5 & -5 & -4 & -2 \\ -3 & 4 & 2 & 6 & 6 \\ -3 & 3 & 0 & 6 & 9 \\ -3 & 1 & -4 & 7 & 8 \\ -3 & 6 & 6 & 6 & 7 \end{bmatrix} \overset{[3;-1/3]}{\longrightarrow} \begin{bmatrix} 3 & -5 & -5 & -4 & -2 \\ -3 & 4 & 2 & 6 & 6 \\ 1 & -1 & 0 & -2 & -3 \\ -3 & 1 & -4 & 7 & 8 \\ -3 & 6 & 6 & 6 & 7 \end{bmatrix} \\ &\overset{[1,3]}{\longrightarrow} \begin{bmatrix} 1 & -1 & 0 & -2 & -3 \\ -3 & 4 & 2 & 6 & 6 \\ 3 & -5 & -5 & -4 & -2 \\ -3 & 1 & -4 & 7 & 8 \\ -3 & 6 & 6 & 6 & 7 \end{bmatrix} \overset{[2,1;3],[3,1;-3],[4,1;3],[5,1;3]}{\longrightarrow} \begin{bmatrix} 1 & -1 & 0 & -2 & -3 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & -2 & -5 & 2 & 7 \\ 0 & -2 & -4 & 1 & -1 \\ 0 & 3 & 6 & 0 & -2 \end{bmatrix} \end{aligned}$$

(i) $[3; -1/3], [1, 3], [2, 1; 3], [3, 1; -3], [4, 1; 3], [5, 1; 3]$

(ii) $P = E(5, 1; 3)E(4, 1; 3)E(3, 1; -3)E(2, 1; 3)E(1, 3)E(3; -1/3) = \begin{bmatrix} 0 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$

(iii) $\det(P) = 1/3$.

(b) Evaluate $\det(A)$. Briefly explain each step.

Solution.

$$|B| = \begin{vmatrix} 1 & 2 & 0 & -3 \\ -2 & -5 & 2 & 7 \\ -2 & -4 & 1 & -1 \\ 3 & 6 & 0 & -2 \end{vmatrix} \overset{[2,1;2],[3,1;2],[4,1;-3]}{\longrightarrow} \begin{vmatrix} 1 & 2 & 0 & -3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 7 \end{vmatrix} = -7.$$

Since $-7 = |B| = |PA| = |P||A| = 1/3 \cdot |A|$, $|A| = -21$.

(c) Write the $(2, 4)$ entry of $\mathrm{adj}(A)$, the adjugate of A, as a determinant. Don't evaluate it.

Solution.

$$\mathrm{adj}(A)_{2,4} = (-1)^{2+4}|A_{4,2}| = \begin{vmatrix} 3 & -5 & -4 & -2 \\ -3 & 2 & 6 & 6 \\ -3 & 0 & 6 & 9 \\ -3 & 6 & 6 & 7 \end{vmatrix}.$$

4. Let $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & 2 & 4 & 1 \\ 1 & -2 & 2 & 5 \end{bmatrix}$. (20 pts)

(a) Explain that A has eigenvalues 6 and 0 without computing the characteristic polynomial of A.
Solution. Since A has constant row sum 6, $A\boldsymbol{v}_1 = 6\boldsymbol{v}_1$, where $\boldsymbol{v}_1 = [1,1,1,1]^T$. Thus 6 is an eigenvalue. Since the first two rows are same, $\det(A) = 0$ and A is not invertible. Hence there exists $A\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution $\boldsymbol{v}_2$. Thus $A\boldsymbol{v}_2 = 0\boldsymbol{v}_2$ and 0 is an eigenvalue. ■

(b) Find all eigenvalues of A.
Solution.

$$\begin{aligned}
|A - xI| &= \begin{vmatrix} 1-x & 2 & 2 & 1 \\ 1 & 2-x & 2 & 1 \\ -1 & 2 & 4-x & 1 \\ 1 & -2 & 2 & 5-x \end{vmatrix} = \begin{vmatrix} 6-x & 2 & 2 & 1 \\ 6-x & 2-x & 2 & 1 \\ 6-x & 2 & 4-x & 1 \\ 6-x & -2 & 2 & 5-x \end{vmatrix} \\
&= (6-x)\begin{vmatrix} 1 & 2 & 2 & 1 \\ 1 & 2-x & 2 & 1 \\ 1 & 2 & 4-x & 1 \\ 1 & -2 & 2 & 5-x \end{vmatrix} = (6-x)\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & -x & 0 & 0 \\ 0 & 0 & 2-x & 0 \\ 0 & -4 & 0 & 4-x \end{vmatrix} \\
&= (6-x)\begin{vmatrix} -x & 0 & 0 \\ 0 & 2-x & 0 \\ -4 & 0 & 4-x \end{vmatrix} = (6-x)(-x)(2-x)(4-x) \\
&= (x-6)x(x-2)(x-4).
\end{aligned}$$

(c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
Solution. $\lambda_1 = 6$: $\boldsymbol{v}_1$ is as in (a).
$\lambda_2 = 0$:

$$A - 0I = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & 2 & 4 & 1 \\ 1 & -2 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 6 & 2 \\ 0 & -4 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

$\lambda_3 = 2$:

$$A - 2I = \begin{bmatrix} -1 & 2 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ -1 & 2 & 2 & 1 \\ 1 & -2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

$\lambda_3 = 4$:

$$A - 4I = \begin{bmatrix} -3 & 2 & 2 & 1 \\ 1 & -2 & 2 & 1 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & -4 & 8 & 4 \\ 1 & -2 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

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