Introduction to Linear Algebra *November 17, 2011*

Final Exam 2011

(Total: 100 pts)

ID#: **Name:**

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by: (15 pts)

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, x_1 - x_2 - x_3, -x_1 + 3x_2 + 5x_3).$$

Let $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ be the standard matrix A for the linear transformation T.

(a) Find A. Show work!

(b) Find $\boldsymbol{v}_1 \times \boldsymbol{v}_2$. Show work!

(c) Let $\boldsymbol{u}_1, \boldsymbol{u}_3, \boldsymbol{u}_3 \in \mathbb{R}^3$. Suppose the volume of the parallelepiped determined by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is 2. What is the volume of the parallelepiped determined by $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)$. Write a brief explanation.

Points:

1.(*a*)	(*b*)	(*c*)	2.(*a*)	(*b*)	(*c*)	(*d*)	(*e*)*	*none*	*	**Total**
								5	10	
3.(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	(*g*)	4.(*a*)*	(*b*)	(*c*)	

2. Let A and B be the following 4×4 matrices, and $\boldsymbol{b} \in \mathbb{R}^4$ given below. (30 pts)

$$A = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}, \quad B = \begin{bmatrix} b & b & b & b \\ b & b & b & b \\ b & b & b & b \\ b & b & b & b \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Find the determinant of A. Show work!

(b) Find the condition that the set of columns of A is linearly dependent. Write a brief explanation.

(c) Find the eigenvalues of A, and their multiplicities.

(d) Suppose $b \neq 0$. Find a linearly independent set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ such that $B\boldsymbol{v}_1 = B\boldsymbol{v}_2 = B\boldsymbol{v}_3 = \mathbf{0}$. Show that it is actually linearly independent.

(e) Let $T = [\boldsymbol{b}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$. (i) Show that T is invertible, and (ii) there is a diagonal matrix D such that $T^{-1}AT = D$.

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. Assume $A\boldsymbol{x} = \boldsymbol{b}$. (35 pts)

$$A = \begin{bmatrix} 3 & -1 & 2 & 0 \\ 1 & 2 & -2 & 5 \\ -1 & 3 & 1 & 1 \\ 2 & 3 & 1 & -2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Briefly explain each step.

(b) Applying the Cramer's rule and express x_2 and x_3 as quotients of determinants. Do not evaluate determinants.

(c) Let $B = \text{adj}(A)$, the adjugate of A. Determine the $(2,3)$-entry of B . Do not evaluate the determinant involved.

Let B be the augmented matrix of $A\boldsymbol{x} = \boldsymbol{b}$. Let C be a matrix obtained from B after applying a series of elementary row operations.

$$B = \begin{bmatrix} 3 & -1 & 2 & 0 & 1 \\ 1 & 2 & -2 & 5 & 2 \\ -1 & 3 & 1 & 1 & 3 \\ 2 & 3 & 1 & -2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -2 & 5 & 2 \\ 0 & 1 & -5 & 12 & 0 \\ 0 & 5 & -1 & 6 & 5 \\ 0 & -7 & 8 & -15 & -5 \end{bmatrix}.$$

(d) Write a sequence of operations applied to B to obtain C using $[i;c]$, $[i,j]$, $[i,j;c]$ notation.

(e) Find a 4×4 matrix P such that $PB = C$.

(f) Express P^{-1} as a product of elementary matrices using the notation $E(i;c)$, $E(i,j)$, $E(i,j;c)$.

(g) Explain that P in (e) is uniquely determined.

4. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3 \in \mathbb{R}^3$ be as follows. (20 pts)

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}.$$

(a) Find the reduced row echelon form of the following matrix. (Show work.)

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{bmatrix}$$

(b) Using (a), explain that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

(c) Find the standard matrix A of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{v}_1) = \boldsymbol{e}_1$, $T(\boldsymbol{v}_2) = \boldsymbol{e}_2$ and $T(\boldsymbol{v}_3) = \boldsymbol{e}_3$.

メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。