Introduction to Linear Algebra *November 18, 2010*

Solutions to Final 2010

(Total: 100 pts)

1. Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be as follows. (10 pts)

$$\boldsymbol{u} = (4, -8, 1),\ \boldsymbol{v} = (2, 1, -2),\ \boldsymbol{w} = (3, -4, 12).$$

(a) The vector $\boldsymbol{p} = \text{proj}_{\boldsymbol{v}}\boldsymbol{u}$ is a scalar multiple of $\boldsymbol{v}$ such that $\boldsymbol{u} - \boldsymbol{p}$ is orthogonal to $\boldsymbol{v}$. Find $\boldsymbol{p}$. (Show work.)

Solution.

$$\begin{aligned} \boldsymbol{p} &= \text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{4\cdot 2 - 8\cdot 1 + 1\cdot(-2)}{2^2 + 1^2 + (-2)^2}(2,1,-2) = -\frac{2}{9}(2,1,-2) \\ &= \left(-\frac{4}{9}, -\frac{2}{9}, \frac{4}{9}\right). \end{aligned}$$

Alternatively, you can set $\boldsymbol{p} = c\boldsymbol{v}$ and determine c by setting $0 = (\boldsymbol{u} - \boldsymbol{p})\cdot\boldsymbol{u}$. So

$$\begin{aligned} 0 &= (\boldsymbol{u} - \boldsymbol{p})\cdot\boldsymbol{v} = (\boldsymbol{u} - c\boldsymbol{v})\cdot\boldsymbol{v} = ((4,-8,1) - c(2,1,-2))\cdot(2,1,-2) \\ &= (4,-8,1)\cdot(2,1,-2) - c(2,1,-2)\cdot(2,1,-2) = -2 - 9c. \end{aligned}$$

Thus $c = -2/9$ and

$$\boldsymbol{p} = c\boldsymbol{v} = -\frac{2}{9}(2,1,-2) = \left(-\frac{4}{9}, -\frac{2}{9}, \frac{4}{9}\right).$$

(b) Compute $\boldsymbol{u}\times\boldsymbol{v}$, and find the volume of the parallelepiped (*heiko-6-mentai*) in 3-space determined by the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. (Show work.)

Solution.

$$\begin{aligned} \boldsymbol{u}\times\boldsymbol{v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & -8 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \left(\begin{vmatrix} -8 & 1 \\ 1 & -2 \end{vmatrix}, -\begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix}, \begin{vmatrix} 4 & -8 \\ 2 & 1 \end{vmatrix} \right) \\ &= (15, 10, 20). \end{aligned}$$

The volume is $|(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}|$. Hence

$$|(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}| = |(15,10,20)\cdot(3,-4,12)| = 15\cdot 3 + 10\cdot(-4) + 20\cdot 12 = 45 - 40 + 240 = 245.$$

2. Evaluate the following determinant. Write explanation in words in detail at each step. (10 pts)

$$\begin{vmatrix} \lambda - c_1 & -c_2 & \cdots & -c_n \\ -c_1 & \lambda - c_2 & \cdots & -c_n \\ \vdots & \vdots & & \vdots \\ -c_1 & -c_2 & \cdots & \lambda - c_n \end{vmatrix}$$

Solution. Let $c = c_1 + c_2 + \cdots + c_n$,

$$\begin{vmatrix} \lambda - c_1 & -c_2 & \cdots & -c_n \\ -c_1 & \lambda - c_2 & \cdots & -c_n \\ \vdots & \vdots & & \vdots \\ -c_1 & -c_2 & \cdots & \lambda - c_n \end{vmatrix} = \begin{vmatrix} \lambda - c & -c_2 & \cdots & -c_n \\ \lambda - c & \lambda - c_2 & \cdots & -c_n \\ \vdots & \vdots & & \vdots \\ \lambda - c & -c_2 & \cdots & \lambda - c_n \end{vmatrix} \tag{1}$$

$$= (\lambda - c)\begin{vmatrix} 1 & -c_2 & \cdots & -c_n \\ 1 & \lambda - c_2 & \cdots & -c_n \\ \vdots & \vdots & & \vdots \\ 1 & -c_2 & \cdots & \lambda - c_n \end{vmatrix} \tag{2}$$

$$= (\lambda - c)\begin{vmatrix} 1 & -c_2 & \cdots & -c_n \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda \end{vmatrix} \tag{3}$$

$$= \lambda^{n-1}(\lambda - c). \tag{4}$$

(1) Column operations $[2, 1; 1], [3, 1; 1], \ldots, [n, 1; 1]$.

(2) Factor out $\lambda - c$ from the first column.

(3) Row operations $[2, 1; -1], [3, 1; -1], \ldots, [n, 1; -1]$.

(4) The determinant of a tridiagonal matrix is the product of the diagonal. One can obtain the same by expanding along the first column successively.

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be the matrices below. Assume $A\boldsymbol{x} = \boldsymbol{b}$.

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 1 & 4 & 12 \\ 2 & -2 & 1 & 1 \\ 2 & 1 & 1 & -3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Briefly explain each step. (10 pts)

Solution.

$$|A| = \begin{vmatrix} 1 & 0 & 2 & 3 \\ -2 & 1 & 4 & 12 \\ 2 & -2 & 1 & 1 \\ 2 & 1 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 8 & 18 \\ 0 & -2 & -3 & -5 \\ 0 & 1 & -3 & -9 \end{vmatrix} = \begin{vmatrix} 1 & 8 & 18 \\ -2 & -3 & -5 \\ 1 & -3 & -9 \end{vmatrix} \tag{5}$$

$$= \begin{vmatrix} 1 & 8 & 18 \\ 0 & 13 & 31 \\ 0 & -11 & -27 \end{vmatrix} = \begin{vmatrix} 1 & 8 & 18 \\ 0 & 2 & 4 \\ 0 & -11 & -27 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ -11 & -27 \end{vmatrix} \tag{6}$$

$$= 2\begin{vmatrix} 1 & 2 \\ -11 & -27 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix} = -10. \tag{7}$$

(5) Row operations $[2, 1; 2], [3, 1; -2], [4, 1; -2]$, and expand along the first row.

(6) Row operations $[2, 1; 2], [3, 1; -1]$, then $[2, 3; 1]$ and expand along the first row.

(7) Factor out 2 from the first row. Row operation [2, 1; 11] and multiply the diagonal of a triangular matrix.

(b) Applying the Cramer's rule and express x_1 and x_4 as quotients of determinants. Do not evaluate determinants. (5 pts)

Solution.

$$x_1 = \frac{\begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 12 \\ -2 & -2 & 1 & 1 \\ 3 & 1 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 & 3 \\ -2 & 1 & 4 & 12 \\ 2 & -2 & 1 & 1 \\ 2 & 1 & 1 & -3 \end{vmatrix}}, \quad x_4 = \frac{\begin{vmatrix} 1 & 0 & 2 & 1 \\ -2 & 1 & 4 & 2 \\ 2 & -2 & 1 & -2 \\ 2 & 1 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 & 3 \\ -2 & 1 & 4 & 12 \\ 2 & -2 & 1 & 1 \\ 2 & 1 & 1 & -3 \end{vmatrix}}$$

(c) Explain that the following system of linear equations with unknowns $y_1, y_2, y_3, y_4, y_5, y_6$ is always consistent and the solution can be written with two free parameters for any a, b, c, d, e, f, g and h. (5 pts)

$$\begin{cases} y_1 & & + 2y_3 & + 3y_4 & + ay_5 & + ey_6 & = 1 \\ -2y_1 & + y_2 & + 4y_3 & + 12y_4 & + by_5 & + fy_6 & = 2 \\ 2y_1 & - 2y_2 & + y_3 & + y_4 & + cy_5 & + gy_6 & = -2 \\ 2y_1 & + y_2 & + y_3 & - 3y_4 & + dy_5 & + hy_6 & = 3 \end{cases}$$

Solution. The augmented matrix B is as follows, where H and H' are the matrices in (d).

$$B = \begin{bmatrix} 1 & 0 & 2 & 3 & a & e & 1 \\ -2 & 1 & 4 & 12 & b & f & 2 \\ 2 & -2 & 1 & 1 & c & g & -2 \\ 2 & 1 & 1 & -3 & d & h & 3 \end{bmatrix} = [A, H, \boldsymbol{b}].$$

Since $\det(A) \neq 0$, A is invertible. Since $A\boldsymbol{x} = \boldsymbol{b}$, $A^{-1}\boldsymbol{b} = \boldsymbol{x}$.

$$A^{-1}B = [A^{-1}A, A^{-1}H, A^{-1}\boldsymbol{b}] = [I, H', \boldsymbol{x}].$$

Since A^{-1} is a product of elementary matrices, after performing corresponding elementary row operations B becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & a' & e' & x_1 \\ 0 & 1 & 0 & 0 & b' & f' & x_2 \\ 0 & 0 & 1 & 0 & c' & g' & x_3 \\ 0 & 0 & 0 & 1 & d' & h' & x_4 \end{bmatrix}$$

This is a reduced row echelon form and rank is 4, no leading 1's in the last column. Thus the system is consistent and the solution can be written with two free parameters.

(d) Let H and H' be as below. Suppose $A^{-1}H = H'$. Explain that the solutions to the system of linear equations in (c) can be expressed as follows, where s and t are free parameters. (5 pts)

$$H = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}, \quad H' = \begin{bmatrix} a' & e' \\ b' & f' \\ c' & g' \\ d' & h' \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \end{bmatrix} - s \cdot \begin{bmatrix} a' \\ b' \\ c' \\ d' \\ -1 \\ 0 \end{bmatrix} - t \cdot \begin{bmatrix} e' \\ f' \\ g' \\ h' \\ 0 \\ -1 \end{bmatrix}.$$

Solution. Using the reduced row echelon form in (c), we have the solution above by setting $x_5 = s$ and $x_6 = t$.

4. Let B be the augmented matrix of a system of linear equations. Let C be a matrix obtained from B after a series of elementary row operation. (25 pts)

$$B = \begin{bmatrix} 1 & 0 & 2 & 3 & a \\ -2 & 1 & 4 & 12 & b \\ 2 & -2 & 1 & 1 & c \\ 2 & 1 & 1 & -3 & d \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 2 & 3 & a' \\ 0 & 1 & -3 & -9 & b' \\ 0 & -2 & -3 & -5 & c' \\ 0 & 1 & 8 & 18 & d' \end{bmatrix}.$$

(a) Express a', b', c' and d' in terms of a, b, c, d. (Show work.)

Solution.

$$\begin{aligned} B &= \begin{bmatrix} 1 & 0 & 2 & 3 & a \\ -2 & 1 & 4 & 12 & b \\ 2 & -2 & 1 & 1 & c \\ 2 & 1 & 1 & -3 & d \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & a \\ 0 & 1 & 8 & 18 & 2a+b \\ 0 & -2 & -3 & -5 & -2a+c \\ 0 & 1 & -3 & -9 & -2a+d \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 0 & 2 & 3 & a \\ 0 & 1 & -3 & -9 & -2a+d \\ 0 & -2 & -3 & -5 & -2a+c \\ 0 & 1 & 8 & 18 & 2a+b \end{bmatrix}, \quad \begin{cases} a' &= a \\ b' &= -2a+d \\ c' &= -2a+c \\ d' &= 2a+b \end{cases} \end{aligned}$$

(b) Write the sequence of operations applied to B to obtain C using $[i;c]$, $[i,j]$, $[i,j;c]$ notation.

Solution. $[2,1;2] \to [3,1;-2] \to [4,1;-2] \to [2,4]$.

(c) Let P be a 4×4 matrix such that $PB = C$. Express each of P and P^{-1} as a product of elementary matrices using the notation $P(i;c)$, $P(i,j)$, $P(i,j;c)$.

Solution.

$$\begin{aligned} P &= P(2,4)P(4,1;-2)P(3,1;-2)P(2,1;2), \\ P^{-1} &= P(2,1;2)^{-1}P(3,1;-2)^{-1}P(4,1;-2)^{-1}P(2,4)^{-1} \\ &= P(2,1;-2)P(3,1;2)P(4,1;2)P(2,4). \end{aligned}$$

(d) Determine P and P^{-1}. (Solution only.)

Solution.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

(e) Explain that P in (c) is uniquely determined.

Solution. The first four columns of B is the matrix A in 3. Since $\det(A) = -10 \neq 0$, it is invertible. Let C' be a 4×4 matrix whose columns are the first four columns of C, then $PA = C'$. So $P = C'A^{-1}$ and P is uniquely determined.

5. Let A, $\boldsymbol{x}$, $\boldsymbol{b}_n$ $(n = 0, 1, 2, \ldots)$ be as follows. (30 pts)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix}, \quad \boldsymbol{b}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{bmatrix} = \boldsymbol{b}_{n+1} = A\boldsymbol{b}_n = A \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}.$$

(a) Find the cofactor matrix $\tilde{A}$, the adjoint matrix $\operatorname{adj}(A)$ and the inverse of A. (Solution only.)

Solution.

$$\tilde{A} = \begin{bmatrix} -5 & -6 & 0 \\ -2 & 0 & -6 \\ 1 & 0 & 0 \end{bmatrix}, \quad \operatorname{adj}(A) = \begin{bmatrix} -5 & -2 & 1 \\ -6 & 0 & 0 \\ 0 & -6 & 0 \end{bmatrix}, \quad A^{-1} = -\frac{1}{6} \begin{bmatrix} -5 & -2 & 1 \\ -6 & 0 & 0 \\ 0 & -6 & 0 \end{bmatrix}.$$

(b) Find the characteristic polynomial and the eigenvalues of A. (Show work.)

Solution.

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & -5 & \lambda - 2 \end{vmatrix} = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ \lambda - 1 & \lambda & -1 \\ \lambda - 1 & -5 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 1) \begin{vmatrix} 1 & -1 & 0 \\ 1 & \lambda & -1 \\ 1 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & \lambda + 1 & -1 \\ 1 & -4 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 1)((\lambda + 1)(\lambda - 2) - 4) = (\lambda - 1)(\lambda^2 - \lambda - 6) \\ &= (\lambda - 3)(\lambda - 1)(\lambda + 2). \end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial and so they are 3, 1 and -2.

(c) Find an eigenvector corresponding to each of the eigenvalues of A. (Show work.)

Solution.

$\lambda = 3$:

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 6 & -5 & 3 - 2 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}.$$

$\lambda = 1$:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 6 & -5 & 1 - 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$\lambda = -2$**:**

$$\begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 6 & -5 & -2-2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}.$$

(d) Find a 3×3 matrix P and a diagonal matrix D such that $AP = PD$. (Give explanation.)

Solution. Let

$$P = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 9 & 1 & 4 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Then

$$\begin{aligned} AP &= A[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = [A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3] = [3\boldsymbol{v}_1, \boldsymbol{v}_2, -2\boldsymbol{v}_3] \\ &= [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = PD. \end{aligned}$$

(e) When $a_0 = 1$, $a_1 = -4$ and $a_2 = -4$, find a_n. (Show work.)

Solution. Since $A\boldsymbol{v}_1 = 3\boldsymbol{v}_1$, $A\boldsymbol{v}_2 = \boldsymbol{v}_2$, $A\boldsymbol{v}_3 = -2\boldsymbol{v}_3$, we have $A^n\boldsymbol{v}_1 = 3^n\boldsymbol{v}_1$, $A^n\boldsymbol{v}_2 = \boldsymbol{v}_2$, $A^n\boldsymbol{v}_3 = (-2)^n\boldsymbol{v}_3$. On the other hand $\boldsymbol{b}_n = A^n\boldsymbol{b}_0$.
Let $\boldsymbol{b}_0 = x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3$. Then

$$\begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = P \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Hence

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -2 & -4 \\ 9 & 1 & 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -5 & -7 \\ 0 & -8 & -5 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 7 \\ 0 & 0 & 15 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $x = -1$, $y = 1$ and $z = 1$, or $\boldsymbol{b}_0 = -\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3$. Now

$$\begin{aligned} \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+1} \end{bmatrix} &= \boldsymbol{b}_n = A^n\boldsymbol{b}_0 = A^n \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} \\ &= A^n(-\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) = -3^n\boldsymbol{v}_1 + \boldsymbol{v}_2 + (-2)^n\boldsymbol{v}_3 \\ &= -3^n \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2)^n \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}. \end{aligned}$$

Therefore $a_n = -3^n + (-2)^n + 1$.

Alternatively, by finding P^{-1} and compute $A^n = PD^nP^{-1}$ can give the following.

$\boldsymbol{b}_n = A^n\boldsymbol{b}_0 = PD^nP^{-1}\boldsymbol{b}_0$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 9 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & (-2)^n \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & \frac{1}{10} & \frac{1}{10} \\ 1 & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{5} & -\frac{4}{15} & \frac{1}{15} \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{5}(-2)^n - \frac{1}{5}3^n + 1 & -\frac{4}{15}(-2)^n + \frac{1}{10}3^n + \frac{1}{6} & \frac{1}{15}(-2)^n + \frac{1}{10}3^n - \frac{1}{6} \\ -\frac{2}{5}(-2)^n - \frac{3}{5}3^n + 1 & \frac{8}{15}(-2)^n + \frac{3}{10}3^n + \frac{1}{6} & -\frac{2}{15}(-2)^n + \frac{3}{10}3^n - \frac{1}{6} \\ \frac{4}{5}(-2)^n - \frac{9}{5}3^n + 1 & -\frac{16}{15}(-2)^n + \frac{9}{10}3^n + \frac{1}{6} & \frac{4}{15}(-2)^n + \frac{9}{10}3^n - \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} \\
&= \begin{bmatrix} (-2)^n - 3^n + 1 \\ -2(-2)^n - 3\,3^n + 1 \\ 4(-2)^n - 9\,3^n + 1 \end{bmatrix}.
\end{aligned}
$$

Therefore $a_n = (-2)^n - 3^n + 1$.

Introduction to Linear Algebra 受講生の皆さんへ

ちょっと難しかったかな。ご苦労様。

明日の夕方には成績を出していると思います。土曜日、月曜日も一日大学にいますが、そのあとは、出張で不在です。試験は、理学館事務室 (N201 レポートボックスの前) で返してもらえるようにします。必ず取りに来て下さい。返却可能になった時点で、NetCommons に案内を出します。成績の表は、subsite のホームページに、出しますので、参照して下さい。詳細を知りたい方は、直接研究室に聞きにきてください。メールや電話では成績の情報はお伝えしません。

一学期間教えて反省点もたくさんありますが、それもホームページに書く予定です。皆さんが書いて下さった、改善点も無論、載せます。

Linear Algebra はこのあと、Linear Algebra (線形代数学) 春学期、Advanced Linear Algebra（線形代数学特論）冬学期につながります。理学部で学ぶと、ここまでが線形代数学で、一年間で学びます。工学部だと、この三つのコースの内容をすこし縮めて、応用をこのあとに加えます。そのあと、数値解析や、線形計画法など、経済や、情報科学でも重要な科目へとつながります。このコースで満足せず、ぜひ、あと 2 コース履修して下さい。物理・化学・情報科学のひとは是非履修して下さい。経済で海外の大学院に留学したい人や、ミクロ経済学関連を専門に勉強したい人も、必須です。

Calculus（微分積分学）が冬学期にあります。Introduction to Linear Algebra の 2 章、3 章で学んだことは出てきます。その微分積分学と、線形代数が統合されて利用されるのが、Advanced Calculus I (解析概論 I) となります。2 年生では、この Advanced Calculus I, II, III と Basic Concepts in Mathematics I, II, III（数学通論 I, II, III）のシリーズが線形代数以外にあります。Advanced Calculus I, II, III は物理の専門科目にも指定されています。化学を学ぶ学生も履修することをお薦めします。Basic Concepts in Mathematics のシリーズは大学の数学入門です。ここからが、大学の数学だと思って下さい。大学の数学を少しは勉強したいという方は、是非履修して下さい。厳密な証明をともなった数学が始まります。特に I は情報科学の学生は是非履修して下さい。

他の分野の専門科目を履修できるのが、ICU の良さです。自分が専門とする分野ではなくても、得意という科目でなくても、是非、数学に挑戦して、大学の数学を味わって下さい。私は次は、数学通論 I で皆さんとお目にかかると思います。よろしく。

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