Linear Algebra I *November 16, 2007*

Solutions to Final 2007

(Toal: 100pts)

1. Find the values of α and β for the following system to have a solution. (10 pts)

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 & = & 6 \\ 6x_1 + 7x_2 + 8x_3 + 9x_4 + 10x_5 & = & 11 \\ 11x_1 + 12x_2 + 13x_3 + 14x_4 + 15x_5 & = & \alpha \\ 16x_1 + 17x_2 + 18x_3 + 19x_4 + 20x_5 & = & \beta \end{cases}$$

Solution. The augmented matrix of the system is as follows.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & 12 & 13 & 14 & 15 & \alpha \\ 16 & 17 & 18 & 19 & 20 & \beta \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1+5 & 2+5 & 3+5 & 4+5 & 5+5 & 6+5 \\ 1+10 & 2+10 & 3+10 & 4+10 & 5+10 & \alpha \\ 1+15 & 2+15 & 3+15 & 4+15 & 5+15 & \beta \end{bmatrix}$$

Applying elementary row operations
$[2,1;-1], [3,1;-1], [4,1;-1], [3,2;-2], [4,2;-3]$,
we have

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & \alpha-16 \\ 0 & 0 & 0 & 0 & 0 & \beta-21 \end{bmatrix} \xrightarrow{[2,1;-5],[2;-1/5],[1,2;-2]} \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & \alpha-16 \\ 0 & 0 & 0 & 0 & 0 & \beta-21 \end{bmatrix}$$

The system has a solution if and only if $\alpha = 16$, $\beta = 21$.

2. Show the following. (10 pts)

$$\begin{vmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{n-1} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-1} \\ 1 & x_3 & {x_3}^2 & \cdots & {x_3}^{n-1} \\ & & \cdots\cdots\cdots & & \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{n-1} \end{vmatrix} = (x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1}) \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-2} \\ 1 & x_3 & {x_3}^2 & \cdots & {x_3}^{n-2} \\ & & \cdots\cdots\cdots & & \\ 1 & x_{n-1} & {x_{n-1}}^2 & \cdots & {x_{n-1}}^{n-2} \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{n-2} & {x_1}^{n-1} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-2} & {x_2}^{n-1} \\ & & & \cdots\cdots\cdots & & \\ 1 & x_{n-1} & {x_{n-1}}^2 & \cdots & {x_{n-1}}^{n-2} & {x_{n-1}}^{n-1} \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{n-2} & {x_n}^{n-1} \end{vmatrix}$$

by subtracting x_n times the $(n-1)$-st column from the n-th column,
or taking the transpose and $[n, n-1; -x_n]$,

$$= \begin{vmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{n-2} & {x_1}^{n-1} - {x_1}^{n-2}x_n \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-2} & {x_2}^{n-1} - {x_2}^{n-2}x_n \\ & & & \cdots\cdots\cdots & & \\ 1 & x_{n-1} & {x_{n-1}}^2 & \cdots & {x_{n-1}}^{n-2} & {x_{n-1}}^{n-1} - {x_{n-1}}^{n-2}x_n \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{n-2} & 0 \end{vmatrix}$$

similarly by subtracting x_n times the $(n-2)$-nd column from the $(n-1)$-st column, etc,,

$$= \begin{vmatrix} 1 & x_1 - x_n & {x_1}^2 - x_1 x_n & \cdots & {x_1}^{n-2} - {x_1}^{n-3} x_n & {x_1}^{n-1} - {x_1}^{n-2} x_n \\ 1 & x_2 - x_n & {x_2}^2 - x_2 x_n & \cdots & {x_2}^{n-2} - {x_2}^{n-3} x_n & {x_2}^{n-1} - {x_2}^{n-2} x_n \\ & & & \cdots\cdots\cdots & & \\ 1 & x_{n-1} - x_n & {x_{n-1}}^2 - x_{n-1} x_n & \cdots & {x_{n-1}}^{n-2} - {x_{n-1}}^{n-3} x_n & {x_{n-1}}^{n-1} - {x_{n-1}}^{n-2} x_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

by expanding along the n-th row,

$$= (-1)^{n+1} \begin{vmatrix} x_1 - x_n & {x_1}^2 - x_1 x_n & \cdots & {x_1}^{n-2} - {x_1}^{n-3} x_n & {x_1}^{n-1} - {x_1}^{n-2} x_n \\ x_2 - x_n & {x_2}^2 - x_2 x_n & \cdots & {x_2}^{n-2} - {x_2}^{n-3} x_n & {x_2}^{n-1} - {x_{n-1}}^{n-2} x_n \\ & & \cdots\cdots\cdots & & \\ x_{n-1} - x_n & {x_{n-1}}^2 - x_{n-1} x_n & \cdots & {x_{n-1}}^{n-2} - {x_{n-1}}^{n-3} x_n & {x_{n-1}}^{n-1} - {x_{n-1}}^{n-2} x_n \end{vmatrix}$$

$$= (-1)^{n+1} \begin{vmatrix} x_1 - x_n & x_1(x_1 - x_n) & \cdots & {x_1}^{n-3}(x_1 - x_n) & {x_1}^{n-2}(x_1 - x_n) \\ x_2 - x_n & x_2(x_2 - x_n) & \cdots & {x_2}^{n-3}(x_2 - x_n) & {x_2}^{n-2}(x_{n-1} - x_n) \\ & & \cdots\cdots\cdots & & \\ x_{n-1} - x_n & x_{n-1}(x_{n-1} - x_n) & \cdots & {x_{n-1}}^{n-3}(x_{n-1} - x_n) & {x_{n-1}}^{n-2}(x_{n-1} - x_n) \end{vmatrix}$$

$$= (-1)^{n+1}(x_1 - x_n)(x_2 - x_n)\cdots(x_{n-1} - x_n) \begin{vmatrix} 1 & x_1 & \cdots & {x_1}^{n-3} & {x_1}^{n-2} \\ 1 & x_2 & \cdots & {x_2}^{n-3} & {x_2}^{n-2} \\ & & \cdots\cdots\cdots & & \\ 1 & x_{n-1} & \cdots & {x_{n-1}}^{n-3} & {x_{n-1}}^{n-2} \end{vmatrix}$$

$$= (x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1}) \begin{vmatrix} 1 & x_1 & \cdots & {x_1}^{n-3} & {x_1}^{n-2} \\ 1 & x_2 & \cdots & {x_2}^{n-3} & {x_2}^{n-2} \\ & & \cdots\cdots\cdots & & \\ 1 & x_{n-1} & \cdots & {x_{n-1}}^{n-3} & {x_{n-1}}^{n-2} \end{vmatrix}.$$

3. Let A be the matrix below. (You can quote the formula of the previous problem.)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \end{bmatrix}$$

(a) Show that A is invertible. (5 pts)

Solution. By applying the formula of the previous problem consecutively,

$$\begin{aligned} |A| &= (5-1)(5-2)(5-3)(5-4) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} \\ &= (5-1)(5-2)(5-3)(5-4)(4-1)(4-2)(4-3) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{vmatrix} \\ &= (5-1)(5-2)(5-3)(5-4)(4-1)(4-2)(4-3)(3-1)(3-2) \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ &= (5-1)(5-2)(5-3)(5-4)(4-1)(4-2)(4-3)(3-1)(3-2)(2-1) \\ &= 4! \cdot 3! \cdot 2! = 288 \neq 0 \end{aligned}$$

Since $|A| \neq 0$, A is invertible.

(b) Find the $(5,1)$-entry of the inverse of A. (5 pts)

Solution. Let $C_{i,j}$ be the cofactor of (i,j)-entry of A. Then

$$(A^{-1})_{5,1} = \frac{C_{1,5}}{|A|} = \frac{1}{|A|}(-1)^{1+5} \begin{vmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 5 & 5^2 & 5^3 \end{vmatrix}$$

$$
\begin{aligned}
&= \frac{1}{|A|}(5-2)(5-3)(5-4)\begin{vmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{vmatrix} \\
&= \frac{1}{|A|}(5-2)(5-3)(5-4)(4-2)(4-3)\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\
&= \frac{1}{|A|}(5-2)(5-3)(5-4)(4-2)(4-3)(3-2) = \frac{12}{288} = \frac{1}{24}.
\end{aligned}
$$

(c) Find the $(1,4)$-entry of the inverse of A. (5 pts)

Solution.

$$
\begin{aligned}
(A^{-1})_{1,4} &= \frac{C_{4,1}}{|A|} = \frac{(-1)^{4+1}}{|A|}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 2^2 & 2^3 & 2^4 \\ 3 & 3^2 & 3^3 & 3^4 \\ 5 & 5^2 & 5^3 & 5^4 \end{vmatrix} = \frac{(-1)^{4+1} 2\cdot 3\cdot 5}{|A|}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 5 & 5^2 & 5^3 \end{vmatrix} \\
&= \frac{(-1)^{4+1} 2\cdot 3\cdot 5}{|A|}(5-1)(5-2)(5-3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{vmatrix} \\
&= \frac{(-1)^{4+1} 2\cdot 3\cdot 5}{|A|}(5-1)(5-2)(5-3)(3-1)(3-2)\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\
&= \frac{(-1)^{4+1} 2\cdot 3\cdot 5}{|A|}(5-1)(5-2)(5-3)(3-1)(3-2)(2-1) = -\frac{30\cdot 48}{288} \\
&= -5.
\end{aligned}
$$

4. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be the matrices below.

$$
A = \begin{bmatrix} 0 & 2 & -5 & 4 \\ -1 & -2 & 0 & 4 \\ 1 & -3 & -1 & 2 \\ 2 & -5 & -3 & 4 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}
$$

(a) Evaluate $\det(A)$. (10 pts)

Solution.

$$
\begin{aligned}
|A| &= \begin{vmatrix} 0 & 2 & -5 & 4 \\ -1 & -2 & 0 & 4 \\ 1 & -3 & -1 & 2 \\ 2 & -5 & -3 & 4 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 & -5 & 2 \\ -1 & -2 & 0 & 2 \\ 1 & -3 & -1 & 1 \\ 2 & -5 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 & -5 & 2 \\ -1 & -2 & 0 & 2 \\ 0 & -5 & -1 & 3 \\ 0 & -9 & -3 & 6 \end{vmatrix} \\
&= 2(-1)(-1)^{2+1}\begin{vmatrix} 2 & -5 & 2 \\ -5 & -1 & 3 \\ -9 & -3 & 6 \end{vmatrix} = 2\cdot(-3)\begin{vmatrix} 2 & -5 & 2 \\ -5 & -1 & 3 \\ 3 & 1 & -2 \end{vmatrix} \\
&= 2\cdot(-3)\begin{vmatrix} 17 & 0 & -8 \\ -2 & 0 & 1 \\ 3 & 1 & -2 \end{vmatrix} = 2\cdot(-3)(-1)^{3+2}\begin{vmatrix} 17 & -8 \\ -2 & 1 \end{vmatrix} = 6.
\end{aligned}
$$

(b) Applying the Cramer's rule to find x_4 of the equation $A\boldsymbol{x} = \boldsymbol{b}$. (10 pts)

Solution.

$$
x_4 = \frac{1}{|A|}\begin{vmatrix} 0 & 2 & -5 & 1 \\ -1 & -2 & 0 & 2 \\ 1 & -3 & -1 & 3 \\ 2 & -5 & -3 & 4 \end{vmatrix} = \frac{1}{|A|}\begin{vmatrix} 0 & 2 & -5 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & -5 & -1 & 5 \\ 0 & -9 & -3 & 8 \end{vmatrix}
$$

$$= \frac{1}{|A|}(-1)(-1)^{2+1}\begin{vmatrix} 2 & -5 & 1 \\ -5 & -1 & 5 \\ -9 & -3 & 8 \end{vmatrix} = \frac{1}{|A|}\begin{vmatrix} 3 & -5 & 1 \\ 0 & -1 & 5 \\ -1 & -3 & 8 \end{vmatrix} = \frac{1}{|A|}\begin{vmatrix} 0 & -14 & 25 \\ 0 & -1 & 5 \\ -1 & -3 & 8 \end{vmatrix}$$

$$= \frac{1}{|A|}(-(-14\cdot 5 - (-1)\cdot 25)) = \frac{45}{6} = \frac{15}{2}.$$

5. Let B, C, $\boldsymbol{x}$ and $\boldsymbol{b}$ be matrices below.

$$B = \begin{bmatrix} 0 & 2 & -5 & 4 & a \\ -1 & -2 & 0 & 4 & b \\ 1 & -3 & -1 & 2 & c \\ 2 & -5 & -3 & 4 & d \end{bmatrix},\ C = \begin{bmatrix} 1 & -3 & -1 & 2 & a' \\ 0 & 1 & -1 & 0 & b' \\ 0 & 2 & -5 & 4 & c' \\ 0 & -5 & -1 & 6 & d' \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

(a) By a consecutive application of elementary row operations, the matrix C is obtained from B. Express a', b', c' and d' in terms of a, b, c, d. (10 pts)

$$B \xrightarrow{[1,3]} \begin{bmatrix} 1 & -3 & -1 & 2 & c \\ -1 & -2 & 0 & 4 & b \\ 0 & 2 & -5 & 4 & a \\ 2 & -5 & -3 & 4 & d \end{bmatrix} \xrightarrow{[2,1;1],[4,1;-2]} \begin{bmatrix} 1 & -3 & -1 & 2 & c \\ 0 & -5 & -1 & 6 & b+c \\ 0 & 2 & -5 & 4 & a \\ 0 & 1 & -1 & 0 & d-2c \end{bmatrix}$$

$$\xrightarrow{[2,4]} \begin{bmatrix} 1 & -3 & -1 & 2 & c \\ 0 & 1 & -1 & 0 & d-2c \\ 0 & 2 & -5 & 4 & a \\ 0 & -5 & -1 & 6 & b+c \end{bmatrix}, \quad \begin{aligned} a' &= c \\ b' &= d-2c \\ c' &= a \\ d' &= b+c. \end{aligned}$$

(b) Let P be a 4×4 matrix such that $PB = C$. Express P^{-1} as a product of elementary matrices using the notation $P(i;\alpha)$, $P(i,j)$, $P(i,j;\beta)$. (10 pts)

Solution. By (a),

$$P = P(2,4)P(4,1;-2)P(2,1;1)P(1,3).$$

Hence

$$\begin{aligned} P^{-1} &= P(1,3)^{-1}P(2,1;1)^{-1}P(4,1;-2)^{-1}P(2,4)^{-1} \\ &= P(1,3)P(2,1;-1)P(4,1;2)P(2,4). \end{aligned}$$

Note that we can find P easily from (a).

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},\ P^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

(c) Show that for any numbers b_1, b_2, b_3, b_4, the equation $B\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions. (10 pts)

Solution. The first four columns of B is the matrix A in the previous problem. Since $|A| \neq 0$, the reduced row echelon form of A is the identity matrix, and that of B has also four leading 1s. Hence $\mathrm{rank}(B) = 4$. Since the augmented matrix of the equation $B\boldsymbol{x} = \boldsymbol{b}$ is of rank 4. Hence it is consistent and it has infinitely many solutions as the number of unknowns is five.

6. Let $A = \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{bmatrix}$

(a) Find the matrices A^2 and A^3. (5 pts)

Solution.

$$A^2 = \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x^2 & 2x & 1 \\ 0 & x^2 & 2x \\ 0 & 0 & x^2 \end{bmatrix},$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} x^2 & 2x & 1 \\ 0 & x^2 & 2x \\ 0 & 0 & x^2 \end{bmatrix} \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x^3 & 3x^2 & 3x \\ 0 & x^3 & 3x^2 \\ 0 & 0 & x^3 \end{bmatrix}.$$

(b) Find the matrix A^n for any natural number $n = 1, 2, 3, \ldots$. (10 pts)

Solution. Since

$$A^4 = \begin{bmatrix} x^3 & 3x^2 & 3x \\ 0 & x^3 & 3x^2 \\ 0 & 0 & x^3 \end{bmatrix} \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x^3 \cdot x & x^3 + 3x^2 \cdot x & 3x^2 + 3x \cdot x \\ 0 & x^3 \cdot x & x^3 + 3x^2 \cdot x \\ 0 & 0 & x^3 \cdot x \end{bmatrix}.$$

So assume the following hold. This is true for $n = 1, 2, 3$ and 4 by the computation carried out above.

$$\begin{aligned} A^n &= \begin{bmatrix} x^{n-1} \cdot x & (n-1)x^{n-1} + x^{n-1} & (1 + 2 + 3 + \cdots + (n-1))x^{n-2} \\ 0 & x^{n-1} \cdot x & (n-1)x^{n-1} + x^{n-1} \\ 0 & 0 & x^{n-1} \cdot x \end{bmatrix} \\ &= \begin{bmatrix} x^n & nx^{n-1} & \frac{n(n-1)}{2}x^{n-2} \\ 0 & x^n & nx^{n-1} \\ 0 & 0 & x^n. \end{bmatrix}. \end{aligned} \tag{1}$$

Then

$$\begin{aligned} A^{n+1} &= \begin{bmatrix} x^n & nx^{n-1} & \frac{n(n-1)}{2}x^{n-2} \\ 0 & x^n & nx^{n-1} \\ 0 & 0 & x^n \end{bmatrix} \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{bmatrix} \\ &= \begin{bmatrix} x^n \cdot x & x^n + nx^{n-1} \cdot x & nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} \cdot x \\ 0 & x^n \cdot x & x^n + nx^{n-1} \cdot x \\ 0 & 0 & x^n \cdot x \end{bmatrix} \\ &= \begin{bmatrix} x^{n+1} & (n+1)x^n & \frac{n(n+1)}{2}x^{n-1} \\ 0 & x^{n+1} & (n+1)x^n \\ 0 & 0 & x^{n+1} \end{bmatrix}. \end{aligned}$$

Since (1) is valid for $n = 4$, it is valid for $n = 5$ and hence for $n = 6, \ldots$. Therefore for all natural numbers (positive integers) we have (1). Formal proof uses mathematical induction.

Hiroshi Suzuki (*hsuzuki@icu.ac.jp*)