BCM I : Final 2017

June 21, 2017

ID#: Name:

1. Let P, Q, R be statements.

   (a) Complete the following truth table. (5 pts)

P	Q	R	$\sim ((P \wedge \sim Q) \wedge \sim R)$	$P \Rightarrow (Q \vee R)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

   (b) Show $\sim ((P \wedge \sim Q) \wedge \sim R) \equiv P \Rightarrow (Q \vee R)$ by using formulas. (5 pts)

2. Let n be a (fixed) positive integer. For $a, b \in \boldsymbol{Z}$, we write $a \equiv b \pmod{n}$, whenever there is an integer c such that $b - a = cn$. Show the following.

   (a) Let $a, b, c, d \in \boldsymbol{Z}$. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

   (i) $a + c \equiv b + d \pmod{n}$, and (5 pts)

   (ii) $ac \equiv bd \pmod{n}$. (5 pts)

1. (10)	2. (20)	3. (10)	4. (20)	5. (20)	6. (20)	Total

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(b) For every positive integer n, $3^{3n+1} + 2^{n+1} \equiv 0 \pmod 5$. (5 pts)

(c) For any integer n, there are integers x and y such that $n = 5x + 7y$. (5 pts)

3. Show that there is an integer m such that for each integer $n \geq m$, there are nonnegative integers a and b such that $n = 5a + 7b$. (10 pts)

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4. Let $f : X \to Y$, $g : Y \to Z$ and $h = g \circ f : X \to Z$ $(x \mapsto g(f(x)))$ be functions. Prove or disprove the following.

   (a) If f is bijective and g is onto, then h is onto. (5 pts)

   (b) If h is onto, then f is onto. (5 pts)

   (c) If f and g are one-to-one, then h is one-to-one. (5 pts)

   (d) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ for all subsets A, B in Y. (5 pts)

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5. For $a, b \in \boldsymbol{R}$ with $a < b$, let $(a, b) = \{x \in \boldsymbol{R} : a < x < b\}$ and $[a, b] = \{x \in \boldsymbol{R} : a \leq x \leq b\}$. Show the following.

   (a) For any $a, b \in \boldsymbol{R}$ with $a < b$, open intervals $(0, 1)$ and (a, b) are numerically equivalent, and closed intervals $[0, 1]$ and $[a, b]$ are numerically equivalent. (5 pts)

   (b) For any $a, b, c, d \in \boldsymbol{R}$ with $a < b$ and $c < d$, open intervals (a, b) and (c, d) are numerically equivalent. (5 pts)

   (c) An open interval $(0, 1)$ and a closed interval $[0, 1]$ are numerically equivalent. (5 pts)

   (d) For any $a, b, c, d \in \boldsymbol{R}$ with $a < b$ and $c < d$, an open interval (a, b) and a closed interval $[c, d]$ are numerically equivalent. (5 pts)

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6. A set A is denumerable if there is a bijection from $\boldsymbol{N}$, the set of positive integers, to A. Let

$$f : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N} \ ((\ell, m) \mapsto 2^{\ell-1}(2m-1)).$$

and show the following.

(a) The function f is one-to-one and onto. (5 pts)

(b) If A and B are denumerable, then $A \times B$ is denumerable. (5 pts)

(c) Let $\boldsymbol{Q}^+$ be the set of positive rational numbers. Then $\boldsymbol{Q}^+$ is denumerable. (10 pts)

Please write your comments:

(1) About this course, especially suggestions for improvements.

(2) Topics in Mathematics or in other subjects you want to study.

BCM I: Solutions to Final 2017

June 21, 2017

1. Let P, Q, R be statements.

   (a) Complete the following truth table.

P	Q	R	$\sim$	$((P$	$\wedge$	$\sim Q)$	$\wedge$	$\sim R)$	P	$\Rightarrow$	$(Q$	$\vee$	$R)$
T	T	T	$\boldsymbol{T}$	T	F	F	F	F	T	$\boldsymbol{T}$	T	T	T
T	T	F	$\boldsymbol{T}$	T	F	F	F	T	T	$\boldsymbol{T}$	T	T	F
T	F	T	$\boldsymbol{T}$	T	T	T	F	F	T	$\boldsymbol{T}$	F	T	T
T	F	F	$\boldsymbol{F}$	T	T	T	T	T	T	$\boldsymbol{F}$	F	F	F
F	T	T	$\boldsymbol{T}$	F	F	F	F	F	F	$\boldsymbol{T}$	T	T	T
F	T	F	$\boldsymbol{T}$	F	F	F	F	T	F	$\boldsymbol{T}$	T	T	F
F	F	T	$\boldsymbol{T}$	F	F	T	F	F	F	$\boldsymbol{T}$	F	T	T
F	F	F	$\boldsymbol{T}$	F	F	T	F	T	F	$\boldsymbol{T}$	F	F	F

   (b) Show $\sim((P\wedge \sim Q)\wedge \sim R) \equiv P \Rightarrow (Q \vee R)$ by using formulas.
   Soln.

$$\begin{aligned}
\sim((P\wedge\sim Q)\wedge\sim R) &\equiv\ \sim(P\wedge\sim Q)\vee(\sim(\sim R)) \qquad \text{(de Morgan)}\\
&\equiv (\sim P\vee\sim(\sim Q))\vee R \qquad \text{(de Morgan and } \sim(\sim X)\equiv X)\\
&\equiv\ \sim P\vee(Q\vee R) \qquad (\sim(\sim X)\equiv X \text{ and a property of } \vee)\\
&\equiv P\Rightarrow(Q\vee R) \qquad (\sim X\vee Y\equiv X\Rightarrow Y)
\end{aligned}$$

2. Let n be a (fixed) positive integer. For $a, b \in \boldsymbol{Z}$, we write $a \equiv b \pmod n$, whenever there is an integer c such that $b - a = cn$. Show the following.

   (a) Let $a, b, c, d \in \boldsymbol{Z}$. If $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then

   (i) $a + c \equiv b + d \pmod n$, and
   Soln. Since $a \equiv b \pmod n$ and $c \equiv d \pmod n$, there exist $s, t \in \boldsymbol{Z}$ such that $b - a = sn$, $d - c = tn$. Therefore,

$$(b + d) - (a + c) = (b - a) + (d - c) = sn + tn = (s + t)n.$$

   Hence $a + c \equiv b + d \pmod n$.

   (ii) $ac \equiv bd \pmod n$.
   Soln. Using the notation in (i),

$$bd - ac = b(d - c) + (b - a)c = tnb + snc = (tb + sc)n.$$

   Hence $ac \equiv bd \pmod n$.

   (b) For every positive integer n, $3^{3n+1} + 2^{n+1} \equiv 0 \pmod 5$.
   Soln. We use (a) repeatedly. Congruences are in modulo 5. Since $3^3 \equiv 2$,

$$3^{3n+1} + 2^{n+1} \equiv 3 \cdot (3^3)^n + 2 \cdot 2^n \equiv 3 \cdot 2^n + 2 \cdot 2^n \equiv 5 \cdot 2^n \equiv 0.$$

   (c) For any integer n, there are integers x and y such that $n = 5x + 7y$.
   Soln. Since $\gcd(5, 7) = 1$, there are $s, t \in \boldsymbol{Z}$ such that $1 = 5s + 7t$. By multiplying n, $n = 5ns + 7nt$. By setting $x = ns$ and $y = nt$, we have $n = 5x + 7y$. For example, $s = 3, t = -2$ and $n = 5 \cdot 3 \cdot n + 7 \cdot (-2) \cdot n$. ■

3. Show that there is an integer m such that for each integer $n \geq m$, there are nonnegative integers a and b such that $n = 5a + 7b$.

   Soln. There are no $a \geq 0$ and $b \geq 0$ satisfying $23 = 5a + 7b$. We know this fact by checking cases $b = 0, 1, 2$. However, $24 = 5 \cdot 2 + 7 \cdot 2$, $25 = 5 \cdot 5 + 7 \cdot 0$, $26 = 5 \cdot 1 + 7 \cdot 3$, $27 = 5 \cdot 4 + 7 \cdot 1$ and $28 = 5 \cdot 0 + 7 \cdot 4$. Let $m = 24$. We claim that for each integer $n \geq m$, there are nonnegative integers a and b such that $n = 5a + 7b$. By our observation above, we may assume that $n \geq 29$. Then $n > n - 5 \geq 24$. Hence by induction hypothesis, there are nonnegative integers a_1 and b_1 such that $n - 5 = 5a_1 + 7b_1$. Let $a = a_1 + 1$ and $b = b_1$. Then $n = 5(a_1 + 1) + 7b_1 = 5a + 7b$. This proves our claim. ■

4. Let $f : X \to Y$, $g : Y \to Z$ and $h = g \circ f : X \to Z$ $(x \mapsto g(f(x)))$ be functions. Prove or disprove the following.

   (a) If f is bijective and g is onto, then h is onto.

   Soln. True. Let $z \in Z$. Since g is onto, there is $y \in Y$ such that $g(y) = z$. Since f is bijective, f is onto. Hence there is $x \in X$ such that $f(x) = y$. Now $h(x) = g(f(x)) = g(y) = z$. Therefore for every $z \in Z$, there is $x \in X$ such that $h(x) = z$ and h is onto. Or, since $f(X) = Y$ and $g(Y) = Z$, $h(X) = g(f(X)) = g(Y) = Z$. ■

   (b) If h is onto, then f is onto.

   Soln. False. Let $X = \{1\}$, $Y = \{1, 2\}$, $Z = \{1\}$, $f(1) = 1$, $g(1) = g(2) = 1$. Then $h(1) = 1$ and h is onto. However, f is not onto, as there is no $x \in X$ such that $f(x) = 2$. ■

   (c) If f and g are one-to-one, then h is one-to-one.

   Soln. True. Suppose $h(x) = h(x')$. Then $g(f(x)) = h(x) = h(x') = g(f(x'))$. Since g is one-to-one, $f(x) = f(x')$. Similarly, since f is one-to-one, $x = x'$. Hence $h(x) = h(x')$ implies $x = x'$ and h is one-to-one. Or, if $x \neq x'$, since f is one-to-one $f(x) \neq f(x')$. Since g is one-to-one, $h(x) = g(f(x)) \neq g(f(x')) = h(x')$. ■

   (d) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ for all subsets A, B in Y.

   Soln. True. Let $x \in f^{-1}(A \cap B)$. Then $f(x) \in A \cap B$. Since $f(x) \in A$, $x \in f^{-1}(A)$. Since $f(x) \in B$, $x \in f^{-1}(B)$. Thus $x \in f^{-1}(A) \cap f^{-1}(B)$. Hence, $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$. Conversely, if $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $x \in f^{-1}(A)$ and hence $f(x) \in A$. Similarly, $x \in f^{-1}(B)$ and hence $f(x) \in B$. Thus $f(x) \in A \cap B$ and $x \in f^{-1}(A \cap B)$. Hence, $f^{-1}(A \cap B) \supseteq f^{-1}(A) \cap f^{-1}(B)$. Therefore, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. ■

5. For $a, b \in \boldsymbol{R}$ with $a < b$, let $(a, b) = \{x \in \boldsymbol{R} : a < x < b\}$ and $[a, b] = \{x \in \boldsymbol{R} : a \leq x \leq b\}$. Show the following.

   (a) For any $a, b \in \boldsymbol{R}$ with $a < b$, open intervals $(0, 1)$ and (a, b) are numerically equivalent, and closed intervals $[0, 1]$ and $[a, b]$ are numerically equivalent.

   Soln. Let $f : (0, 1) \to (a, b)$ $(x \mapsto (b - a)x + a)$. Then f is strictly increasing and onto. Hence f is bijective and open intervals $(0, 1)$ and (a, b) are numerically equivalent. Similarly, Let $f : [0, 1] \to [a, b]$ $(x \mapsto (b - a)x + a)$. Then f is strictly increasing and onto. Hence f is bijective and open intervals $[0, 1]$ and $[a, b]$ are numerically equivalent. ■

   (b) For any $a, b, c, d \in \boldsymbol{R}$ with $a < b$ and $c < d$, open intervals (a, b) and (c, d) are numerically equivalent.

   Soln. Since $a, b \in \boldsymbol{R}$ with $a < b$ are arbitrary, $(0, 1)$ and (c, d) are numerically equivalent. Since numerical equivalence is an equivalence relation, $(a, b) \sim (0, 1) \sim (c, d)$ implies $(a, b) \sim (c, d)$ and (a, b) and (c, d) are numerically equivalent. ■

(c) An open interval $(0,1)$ and a closed interval $[0,1]$ are numerically equivalent.
Soln. $f : (0,1) \to [0,1]$ $(x \mapsto x)$ is one-to-one. Hence $|(0,1)| \le |[0,1]|$. By (a), $[0,1]$ and $[1/4, 3/4]$ are numerical equivalent. Let g be a bijection from $[0,1]$ to $[1/4, 3/4]$. Then $h : [0,1] \to (0,1)$ $(x \mapsto g(x))$ is one-to-one. Note that $g([0,1]) = [1/4, 3/4] \subseteq (0,1)$. Hence $|[0,1]| \le |(0,1)|$. Therefore, by Schröder-Bernstein's Theorem, $(0,1)$ and $[0,1]$ are numerically equivalent. ■

(d) For any $a, b, c, d \in \boldsymbol{R}$ with $a < b$ and $c < d$, an open interval (a,b) and a closed interval $[c,d]$ are numerically equivalent.
Soln. By (a), (a,b) is numerically equivalent to $(0,1)$. By (c), $(0,1)$ is numerically equivalent to $[0,1]$. By (a), $[0,1]$ is numerically equivalent to $[c,d]$. Since numerical equivalence is an equivalence relation, (a,b) is numerically equivalent to $[c,d]$. ■

6. A set A is denumerable if there is a bijection from $\boldsymbol{N}$, the set of positive integers, to A. Let
$$f : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N} \ ((\ell, m) \mapsto 2^{\ell-1}(2m-1)).$$
and show the following.

(a) The function f is one-to-one and onto.
Soln. Suppose $f(\ell, m) = f(\ell', m')$. Then by definition, $2^{\ell-1}(2m-1) = 2^{\ell'-1}(2m'-1)$. Suppose $\ell < \ell'$. Then $2m - 1 = 2^{\ell'-\ell}(2m'-1)$ and the left hand side is odd and the right hand side is even, which is absurd. Hence $\ell = \ell'$ and $2m-1 = 2m'-1$. Now $m = m'$. Therefore, if $f(\ell, m) = f(\ell', m')$, then $(\ell, m) = (\ell', m')$ and f is one-to-one. Let n be a positive integer. Let $2^{\ell-1}$ is the highest power of 2 dividing n. Then ℓ is a positive integer, $n = 2^{\ell-1}n'$ and n' is a nonnegative odd integer. Hence there is a positive integer m such that $n' = 2m - 1$. Hence $n = 2^{\ell-1}(2m-1) = f(\ell, m)$ and f is onto. ■

(b) If A and B are denumerable, then $A \times B$ is denumerable.
Soln. Let $f^{-1} : \boldsymbol{N} \to \boldsymbol{N} \times \boldsymbol{N}$ $(n \mapsto f^{-1}(n) = (f_1(n), f_2(n)))$ be the inverse function of f in (a). Since both A and B are denumerable, there are bijections $g : \boldsymbol{N} \to A$ and $h : \boldsymbol{N} \to B$. Let
$$k : N \to A \times B \ (n \mapsto (g(f_1(n)), h(f_2(n))).$$
We claim that k is bijective. If
$$(g(f_1(n)), h(f_2(n))) = k(n) = k(n') = (g(f_1(n')), h(f_2(n'))),$$
then $g(f_1(n)) = g(f_1(n'))$ and $h(f_2(n)) = h(f_2(n'))$. Since g and h are one-to-one, $(f_1(n), f_2(n)) = (f_1(n'), f_2(n'))$. Since f^{-1} is bijective and $f^{-1}(n) = (f_1(n), f_2(n)) = (f_1(n'), f_2(n')) = f^{-1}(n')$, $n = n'$. Hence k is one-to-one. Let $(a,b) \in A \times B$. Since g and h are bijective, there are $m, m' \in \boldsymbol{N}$ such that $g(m) = a$ and $h(m') = b$. Since f^{-1} is bijective, there is $n \in \boldsymbol{N}$ such that $f^{-1}(n) = (f_1(n), f_2(n)) = (m, m')$. Now $k(n) = (g(f_1(m)), h(f_2(m'))) = (g(m), h(m')) = (a,b)$ and k is onto. Therefore, $A \times B$ is denumerable. ■

(c) Let $\boldsymbol{Q}^+$ be the set of positive rational numbers. Then $\boldsymbol{Q}^+$ is denumerable.
Soln. Since $\boldsymbol{N} \subseteq \boldsymbol{Q}^+$, $j : \boldsymbol{N} \to \boldsymbol{Q}^+$ $(x \mapsto x)$ is one-to-one and $|\boldsymbol{N}| \le |\boldsymbol{Q}^+|$. For each $x \in \boldsymbol{Q}^+$, write $x = m/n$ with $m, n \in \boldsymbol{N}$ and $\gcd(m,n) = 1$. Then (m,n) is uniquely determined. Since m and n are determined uniquely from x, write $m = m(x)$ and $n = n(x)$. Then $p : \boldsymbol{Q}^+ \to \boldsymbol{N} \times \boldsymbol{N}$ $(x \mapsto (m(x), n(x)))$ is a one-to-one function. Hence $f \circ p : \boldsymbol{Q}^+ \to \boldsymbol{N}$ $(x \mapsto f(m(x), n(x)))$ is a composition of two one-to-one functions, it is one-to-one by 4 (c). Thus $|\boldsymbol{Q}^+| \le |\boldsymbol{N}|$. By Schröder-Bernstein's Theorem, $|\boldsymbol{Q}^+| = |\boldsymbol{N}|$, and $\boldsymbol{Q}^+$ is denumerable. ■