Algebra III Final (Example)[1]

1. Let R be a commutative ring. Suppose that the ideals of R are only $\{0\}$ and R. Show that R is a field.

2. Let L be an extension field of K.

   (a) Let $\alpha \in L$. Write the definition that α is algebraic over K.

   (b) Show that all elements of K are algebraic over K.

   (c) Give an element of the complex number field $\boldsymbol{C}$ that is algebraic over the rational number field $\boldsymbol{Q}$ but not in $\boldsymbol{Q}$, and show that it is actually algebraic over $\boldsymbol{Q}$.

3. Let $L = \boldsymbol{Q}(\sqrt{5}, \sqrt{7})$ be a subfield of $\boldsymbol{C}$.

   (a) Show that L is a simple extension of $\boldsymbol{Q}$, i.e., there is an element $\alpha \in L$ such that $L = \boldsymbol{Q}(\alpha)$.

   (b) Show that $(L : \boldsymbol{Q}) = 4$.

   (c) Show that all elements of L are algebraic over $\boldsymbol{Q}$.

   (d) Let σ be an automorphism of L.

      i. Show that $\sigma(a) = a$ for all $a \in \boldsymbol{Q}$.

      ii. Show that either $\sigma(\sqrt{5}) = \sqrt{5}$ or $\sigma(\sqrt{5}) = -\sqrt{5}$

   (e) Show that there is an automorphism τ of $\boldsymbol{Q}(\sqrt{5})$ such that $\tau(\sqrt{5}) = -\sqrt{5}$.

   (f) Show that there is an automorphism σ of L such that $\sigma(\sqrt{5}) = -\sqrt{5}$ and $\sigma(\sqrt{7}) = \sqrt{7}$.

   (g) Show that L is a normal extension of $\boldsymbol{Q}$.

   (h) Determine $\mathrm{Gal}(L/\boldsymbol{Q})$.

   (i) Determine all subfields of L.

4. Let $K = \boldsymbol{Z}/2\boldsymbol{Z}$ be a field with two elements.

   (a) Let $f \in K[t]$.

      i. Write the definition that f is irreducible over K.

      ii. Find all irreducible polynomials in $K[t]$ of degree at most four and show that they are actually all irreducible polynomials of degree at most four.

   (b) Show that there is an extension field L of K with 16 elements. Let L be such a field in the following problems.

   (c) Show that $x + x = 0$ for all $x \in L$.

   (d) Let $\alpha : L \to L$ $(x \mapsto x^2)$. Show that α is an automorphism of L.

   (e) Determine $\mathrm{Gal}(L/K)$.

   (f) Determine all subfields of L.

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[1]This is based on Final 2002