Algebra I: Midterm 2017

May 31, 2017

ID#: **Name:** (10 pts each)

1. Let H and K be subgroups of a group G. Let $a, b \in G$. Show the following.

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.

   (b) $HH = H^{-1} = H$.

   (c) $aHa^{-1} \cap K$ is a subgroup of K.

   (d) If $aH \subseteq bK$, then $H \leq K$.

2. Let H and K be subgroups of a group G. Show the following.

(a) Suppose $xhx^{-1} \in H$ for all $x \in G$ and $h \in H$. State the definition of a normal subgroup and show that H is a normal subgroup of G.

(b) If K is of prime order, i.e., $|K| = p$, where p is a prime number, then $K \approx \boldsymbol{Z}_p$. (Show both K is cyclic and it is isomorphic to $\boldsymbol{Z}_p$.)

(c) Suppose both H and K are normal subgroups of G and $|H| = p$, $|K| = q$, where p and q are distinct primes. If $G = HK$, then G is cyclic.

3. Let $G = H \oplus K$, $H = \boldsymbol{Z}_{15}$ and $K = \boldsymbol{Z}_9$.

   (a) Find all subgroups of H. For each subgroup of H, list all generators.

   (b) Find the number of elements of order 9 in G. Show work.

   (c) Find all subgroups of order 9. Show work.

Algebra I: Solutions to Midtern 2017

May 31, 2017

1. Let H and K be subgroups of a group G. Let $a, b \in G$. Show the following.

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.

   Soln. Since $H \leq G$, $H \neq \emptyset$. Let $a \in H$. Then $a^{-1} \in H$ and $e = aa^{-1} \in H$.

   Suppose $aH = bH$. Since $e \in H$, $aH = bH$ implies that $b = be \in bH = aH$. Hence there exists $h \in H$ such that $b = ah$. Therefore, by multiplying a^{-1} to both hand sides from the left, $a^{-1}b = h \in H$.

   Conversely let $a^{-1}b = h \in H$. Then $b = ah$ and

   $$bH = ahH \subseteq aH = aeH = ahh^{-1}H = aa^{-1}bh^{-1}H \subseteq bH.$$

   Therefore $aH = bH$. ■

   (b) $HH = H^{-1} = H$.

   Soln. Since H is a subgroup of G, the identity element $e \in H$, for all $x, y \in H$, $xy \in H$ and $x^{-1} \in H$. Moreover, $x = (x^{-1})^{-1} \in H^{-1}$. Thus, $H \subseteq H^{-1}$ and

   $$H = eH \subseteq HH \subseteq H \subseteq H^{-1} \subseteq H.$$

   Therefore, $HH = H^{-1} = H$. ■

   (c) $aHa^{-1} \cap K$ is a subgroup of K.

   Soln. Let $L = aHa^{-1} \cap K$. Clearly, $e = aea^{-1} \in aHa^{-1} \cap K = L$ and $L \neq \emptyset$. Let $x, x' \in L$. Since $L = aHa^{-1} \cap K$, $x, x' \in K$ and there exist $h, h' \in H$ such that $x = aha^{-1}$ and $y = ah'a^{-1}$. Since K is a subgroup of G, $xx'^{-1} \in K$. Moreover, $xx'^{-1} = aha^{-1}ah'^{-1}a^{-1} = ahh'^{-1}a^{-1} \in aHa^{-1}$. Hence $xx^{-1} \in aHa^{-1} \cap K = L$. Therefore, by the one step subgroup test, L is a subgroup of K. ■

   (d) If $aH \subseteq bK$, then $H \leq K$.

   Soln. Since $a = ae \in aH \subseteq bK$, $aK = bK$. Note that the condition implies $a^{-1}b = (b^{-1}a)^{-1} \in K$. Use 1 (a). Hence $aH \subseteq aK$ and we have $H \subseteq K$. ■

2. Let H and K be subgroups of a group G. Show the following.

   (a) Suppose $xhx^{-1} \in H$ for all $x \in G$ and $h \in H$. State the definition of a normal subgroup and show that H is a normal subgroup of G.

   Soln. A subgroup H is a normal subgroup of G if and only if $aHa^{-1} = H$ for all $a \in G$.

   It suffices to show that $H \subseteq aHa^{-1}$ for all $a \in G$, which is equivalent to $a^{-1}Ha \subseteq H$. Let $h \in H$ and $a \in G$. Then $a^{-1} \in G$ and hence $a^{-1}xa = a^{-1}x(a^{-1})^{-1} \in H$. Therefore, $a^{-1}Ha \subseteq H$ and H is a normal subgroup of G. ■

(b) If K is of prime order, i.e., $|K| = p$, where p is a prime number, then $K \approx \boldsymbol{Z}_p$. (Show both K is cyclic and it is isomorphic to $\boldsymbol{Z}_p$.)

Soln. Since a prime number is at least 2, there is a nonidentity element $x \in K$. Then $\langle x \rangle$ is a subgroup of K of order at least 2. By Lagrange's Theorem, $|\langle x \rangle|$ divides $p = |K|$. Hence $|\langle x \rangle| = p$ and $\langle x \rangle = K$ as $\langle x \rangle \subseteq K$ by our choice of x. Thus K is a cyclic group of order p. Let

$$\phi : \boldsymbol{Z}_p = \{0, 1, \ldots, p-1\} \to K = \langle x \rangle = \{e, x, x^2, \ldots, x^{p-1}\} \ (n \mapsto x^n).$$

Then ϕ is a bijection and $\phi(m+n) = x^{m+n} = x^m x^n = \phi(m)\phi(n)$ and ϕ is operation preserving. Note that $x^s = e$ if and only if $p \mid s$ for every integer s and $m + n \in \boldsymbol{Z}_p$ is computed modulo p. ■

(c) Suppose both H and K are normal subgroups of G and $|H| = p$, $|K| = q$, where p and q are distinct primes. If $G = HK$, then G is cyclic.

Soln. Since $H \cap K$ is a subgroup of H and K, the order of $H \cap K$ divides p and q. Hence it is one. Thus $H \cap K = \{e\}$. Since $G = HK$, $G = H \times K$. Since $G = H \times K \approx H \oplus K \approx \boldsymbol{Z}_p \oplus \boldsymbol{Z}_q = \boldsymbol{Z}_{pq}$, G is cyclic. Note that p and q are coprime to each other, $\boldsymbol{Z}_p \oplus \boldsymbol{Z}_q = \boldsymbol{Z}_{pq}$. ■

3. Let $G = H \oplus K$, $H = \boldsymbol{Z}_{15}$ and $K = \boldsymbol{Z}_9$.

(a) Find all subgroups of H. For each subgroup of H, list all generators.

Soln. Since H is cyclic, every subgroup of H is cyclic. Moreover, for each divisor of the order, there exists a subgroup of its order, we have the following.

i. Order 1: $\{0\}$, 0 is the only generator.
ii. Order 3: $\{0, 5, 10\}$, 5 and 10 are generators.
iii. Order 5: $\{0, 3, 6, 9, 12\}$, $3, 6, 9, 12$ are generators.
iv. Order 15: H: $1, 2, 4, 7, 8, 11, 13, 14$ are generators.

■

(b) Find the number of elements of order 9 in G. Show work.

Soln. Let H_1 be $\{0, 5, 10\}$ and $K_1 = \{0, 3, 6\}$ the subgroups of order 3 in H and K. Then all elements of order 9 are contained in $H_1 \oplus K$ and all elements of order 1 and 3 are in $H_1 \oplus K_1$. Therefore, elements of order 9 in G are in $H_1 \oplus K \setminus H_1 \oplus K_1$. Hence, there are $27 - 9 = 18$ in all. ■

(c) Find all subgroups of order 9. Show work.

Soln. If the subgroup is cyclic, it is generated by an element of order 9 and each cyclic subgroup of order 9 contains exactly 6 elements of order 9. Hence there are $18/6 = 3$ cyclic subgroups. If it is not cyclic, every nonidentity element is of order 3. Therefore it is contained in $H_1 \oplus K_1$ in the previous problem. Since it is of order 9, there are four subgroups of order 9 in all. ■