Algebra I: Final 2018

June 20, 2018

ID#: **Name:**

Quote the following when necessary.

A. Subgroup H of a group G:

$$H \leq G \Leftrightarrow \emptyset \neq H \subseteq G,\ xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

B. Order of an Element: Let g be an element of a group G. Then $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$ is a subgroup of G. If there is a positive integer m such that $g^m = e$, where e is the identity element of G, $|g| = \min\{m \mid g^m = e,\ m \in \mathbb{N}\}$ and $|g| = |\langle g \rangle|$. Moreover, for any integer n, $|g|$ divides n if and only if $g^n = e$.

C. Lagrange's Theorem: If H is a subgroup of a finite group G, then $|G| = |G : H||H|$.

D. Normal Subgroup: A subgroup H of a group G is normal if $gHg^{-1} = H$ for all $g \in G$. If H is a normal subgroup of G, then G/H becomes a group with respect to the binary operation $(gH)(g'H) = gg'H$.

E. Direct Product: If $\gcd(m, n) = 1$, then $\mathbb{Z}_{mn} \approx \mathbb{Z}_m \oplus \mathbb{Z}_n$ and $U(mn) \approx U(m) \oplus U(n)$.

F. Isomorphism Theorem: If $\alpha : G \to \overline{G}$ is a group homomorphism, $\mathrm{Ker}(\alpha) = \{x \in G \mid \alpha(x) = e_{\overline{G}}\}$, where $e_{\overline{G}}$ is the identity element of $\overline{G}$. Then $G/\mathrm{Ker}(\alpha) \approx \alpha(G)$.

G. Sylow's Theorem: For a finite group G and a prime p, let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G. Then $\mathrm{Syl}_p(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. Then $|\mathrm{Syl}_p(G)| = |G : N(P)| \equiv 1 \pmod p$, where $N(P) = \{x \in G \mid xPx^{-1} = P\}$.

Other Theorems: List other theorems you applied in your solutions.

Please write your message: Comments on group theory. Suggestions for improvements of this course. Write on the back of this sheet is also welcome.

1. Let H and K be subgroups of a group G. Let $a, b \in G$. Show the following. (20 pts)

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.

   (b) $HH^{-1} = H$.

   (c) If $HK \leq G$, then $HK = KH$.

   (d) If $HK = KH$, then $HK \leq G$.

2. Let G be a group, H a subgroup of G, N a normal subgroup of G, $\phi : G \to G/N \, (x \mapsto xN)$ a function from G to a factor group G/N. Show the following. (20 pts)

   (a) ϕ is an onto group homomorphism.

   (b) $\phi(H)$ is a subgroup of G/N.

   (c) Let $\bar{H} = \phi(H)$. Then $\phi^{-1}(\bar{H}) = HN$.

   (d) $H/(N \cap H) \approx HN/N$.

3. Answer the following questions on Abelian groups of order $24 = 2^3 \cdot 3$. (20 pts)

   (a) Using the Fundamental Theorem of Finite Abelian Groups, list all non-isomorphic Abelian groups of order 24. Express each as an external direct product of cyclic groups.

   (b) What is the largest order of the elements of $U(35)$? Show work.

   (c) Express $U(35)$ as an internal direct product of cyclic subgroups.

   (d) Is $U(35)$ isomorphic to $U(72)$? Show work.

4. Let p be a prime number, and let $G = \left\{ \begin{bmatrix} a & bp \\ b & a \end{bmatrix} \middle| \ a, b \in \mathbb{Q}, a \neq 0 \text{ or } b \neq 0 \right\}$. Let $\mathbb{R}^*$ be the multiplicative group of nonzero real numbers. Show the following. (20 pts)

(a) G is closed under matrix product, i.e., if $A, B \in G$, then $AB \in G$.

(b) G is a group with respect to matrix product.

(c) Suppose $\phi : G \to \mathbb{R}^* \left(\begin{bmatrix} a & bp \\ b & a \end{bmatrix} \mapsto a + b\alpha \right)$ is a group homomorphism for some $\alpha \in \mathbb{R}^*$. Then $\alpha = \pm\sqrt{p}$.

(d) $G \approx H = \{z \in \mathbb{R}^* \mid z = x + y\sqrt{p} \text{ for some } x, y \in \mathbb{Q}\} \leq \mathbb{R}^*$.

5. Let G be a group of order $1225 = 5^2 \cdot 7^2$. Let $P \in \mathrm{Syl}_5(G)$ and $Q \in \mathrm{Syl}_7(G)$. Show the following. (20 pts)

   (a) P is a normal subgroup of G.

   (b) P is Abelian.

   (c) $G = P \times Q$.

   (d) G has an element of order 35.

Algebra I: Solutions to Final 2018

June 20, 2018

1. Let H be a subgroup of a group G. Let $a, b \in G$. Show the following. (20 pts)

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.
   Soln. Since $H \leq G$, $H \neq \emptyset$. Let $a \in H$. Then, $a^{-1} \in H$ and $e = aa^{-1} \in H$.
   Suppose $aH = bH$. Since $e \in H$, $aH = bH$ implies that $b = be \in bH = aH$. Hence, there exists $h \in H$ such that $b = ah$. Therefore, by multiplying a^{-1} to both hand sides from the left, $a^{-1}b = h \in H$.
   Conversely, let $a^{-1}b = h \in H$. Then, $b = ah$ and
   $$bH = ahH \subseteq aH = aeH = ahh^{-1}H = aa^{-1}bh^{-1}H \subseteq bH.$$
   Therefore, $aH = bH$. ■

   (b) $HH^{-1} = H$.
   Soln. Since H is a subgroup,
   $$H = (H^{-1})^{-1} \subseteq H^{-1} = eH^{-1} \subseteq HH^{-1} \subseteq H.$$
   Therefore $H = H^{-1} = HH^{-1}$. ■

   (c) If $HK \leq G$, then $HK = KH$.
   Soln. Since HK, H and K are subgroups of G, by the proof of (b),
   $$HK = (HK)^{-1} = K^{-1}H^{-1} = KH.$$
   ■

   (d) If $HK = KH$, then $HK \leq G$.
   Soln. We use the Two Step Subgroup Test. By the proof of (b),
   $$HKHK = HHKK = HK, \quad (HK)^{-1} = K^{-1}H^{-1} = KH = HK.$$
   Hence, $HK \leq G$. ■

2. Let G be a group, H a subgroup of G, N a normal subgroup of G, $\phi : G \to G/N (x \mapsto xN)$ a function from G to a factor group G/N. Show the following. (20 pts)

   (a) ϕ is an onto group homomorphism.
   Soln. Let $x, y \in G$. Since $(xN)(yN) = xyN$, we have $\phi(xy) = xyN = (xN)(yN) = \phi(x)\phi(y)$. Therefore, ϕ is operation preserving. Since every element of G/N has a form xN with $x \in G$, $xN = \phi(x)$ and ϕ is onto. ■

   (b) $\phi(H)$ is a subgroup of G/N.
   Soln. First, note that N is the identity element in G/N, $\phi(e) = eN = N$ and $\phi(x^{-1}) = x^{-1}N = (xN)^{-1} = \phi(x)^{-1}$. Let $\phi(h), \phi(h') \in \phi(H)$ with $h, h' \in H$. Then $\phi(h)\phi(h')^{-1} = \phi(h)\phi(h'^{-1}) = \phi(hh'^{-1}) \in \phi(H)$. Therefore, by the One-Step-Subgroup Test, $\phi(H) \leq G/N$. ■

(c) Let $\bar{H} = \phi(H)$. Then $\phi^{-1}(\bar{H}) = HN$.

Soln. Let $hn \in HN$ with $h \in H$ and $n \in N$. Then, $\phi(hn) = hnN = hN \in \phi(H) = \bar{H}$ by 1(a), as $(hn)^{-1}h = n^{-1}h^{-1}h = n^{-1} \in N$. Hence, $HN \subseteq \phi^{-1}(\bar{H})$. If $x \in \phi^{-1}(\bar{H})$, $\phi(x) \in \bar{H} = \phi(H)$. Hence, there exists $h \in H$ such that $\phi(x) = \phi(h)$. Then $xN = hN$ and $h^{-1}x \in N$ by 1 (a). Let $h^{-1}x = n \in N$. Then $x = hn \in HN$ and $\phi^{-1}(\bar{H}) \subseteq HN$. Therefore, $\phi^{-1}(\bar{H}) = HN$. ■

(d) $H/(N \cap H) \approx HN/N$.

Soln. Since $\phi(H) = \phi(HN) = HN/N$ as $N \lhd HN$ and $\text{Ker}(\phi) \cap H = H \cap N$, $H/H \cap N \approx HN/N$ by the First Isomorphism Theorem. Here, the theorem is applied to the group homomorphism ϕ restricted to H. ■

3. Answer the following questions on Abelian groups of order $24 = 2^3 \cdot 3$. (20 pts)

(a) Using the Fundamental Theorem of Finite Abelian Groups, list all non-isomorphic Abelian groups of order 24. Express each as an external direct product of cyclic groups.

Soln. There are three isomorphism classes.

$$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \approx \mathbb{Z}_{24}, \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{12}, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6. \quad ■$$

(b) What is the largest order of the elements of $U(35)$? Show work.

Soln. By E,

$$U(35) \approx U(5) \oplus U(7) \approx \mathbb{Z}_4 \oplus \mathbb{Z}_6 \approx \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \quad \text{as } U(5) = \langle 2 \rangle,\ U(7) = \langle 3 \rangle.$$

Hence $U(35)$ is isomorphic to the second type in (b) and the largest order of the elements of $U(35)$ is 12. ■

(c) Express $U(35)$ as an internal direct product of cyclic subgroups.

Soln. Since none of $2, 4, 8, 16, 32 = 3, 6$ is 1, the order of 2 in $U(35)$ is 12 and the unique element of order 2 in $\langle 2 \rangle$ is 6 and not 34, $U(35) = \langle 34 \rangle \times \langle 2 \rangle \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$. ■

(d) Is $U(35)$ isomorphic to $U(72)$? Show work.

Soln. Since every nonidentity element of $U(8) = \{1, 3, 5, 7, 9\}$ is of order 2, $U(72) \approx U(8) \oplus U(9) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6 \not\approx U(35)$. ■

4. Let p be a prime number, and let $G = \left\{ \begin{bmatrix} a & bp \\ b & a \end{bmatrix} \,\middle|\, a, b \in \mathbb{Q}, a \neq 0 \text{ or } b \neq 0 \right\}$. Let $\mathbb{R}^*$ be the multiplicative group of nonzero real numbers. Show the following. (20 pts)

(a) G is closed under matrix product, i.e., if $A, B \in G$, then $AB \in G$.

Soln. Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & p \\ 1 & 0 \end{bmatrix}$. Then $J^2 = pI$ and $G = \{aI + bJ \mid a, b \in \mathbb{Q}, a \neq 0 \text{ or } b \neq 0\}$. Let $A = aI + a'J$ and $B = bI + b'J$. Since $\det(A) = a^2 - a'^2 p$ and $\sqrt{p} \notin \mathbb{Q}$, $\det(A) \neq 0$ if and only if $a \neq 0$ or $a' \neq 0$. Similarly, $\det(B) \neq 0$. Thus $\det(AB) = \det(A)\det(B) \neq 0$ and

$$AB = (aI + a'J)(bI + b'J) = (ab + a'b'p)I + (ab' + a'b)J \in G.$$

Therefore, G is closed under matrix product. ■

(b) G is a group with respect to matrix product.

Soln. For $A = \begin{bmatrix} a & bp \\ b & a \end{bmatrix}$, $A^{-1} = \dfrac{1}{a^2 - bp^2} \begin{bmatrix} a & -bp \\ -b & a \end{bmatrix} \in G$ and G is a subgroup of $\mathrm{GL}(2, \mathbb{R})$ by (a) and the Two-Step-Subgroup Test. ■

(c) Suppose $\phi : G \to \mathbb{R}^* \left(\begin{bmatrix} a & bp \\ b & a \end{bmatrix} \mapsto a + b\alpha \right)$ is a group homomorphism for some $\alpha \in \mathbb{R}^*$. Then $\alpha = \pm\sqrt{p}$.

Soln. Since $J^2 = pI$, $\phi(J)^2 = \phi(J^2) = \phi(pI) = p \in \mathbb{R}^*$. Hence a nonzero real number $\alpha = \phi(J)$ has to be $\pm\sqrt{p}$. ■

(d) $G \approx H = \{z \in \mathbb{R}^* \mid z = x + y\sqrt{p} \text{ for some } x, y \in \mathbb{Q}\} \le \mathbb{R}^*$.

Soln. Let $\psi : G \to \mathbb{R}^* \left(\begin{bmatrix} a & bp \\ b & a \end{bmatrix} \mapsto a + b\sqrt{p} \right)$. Then using the notation in (a),

$$\begin{aligned} \psi(AB) &= \psi((ab + a'b'p)I + (ab' + a'b)J) = (ab + a'b'p) + (ab' + a'b)\sqrt{p} \\ &= (a + a'\sqrt{p})(b + b'\sqrt{p}) = \psi(A)\psi(B). \end{aligned}$$

Hence ψ is a group homomorphism. By the definition of ψ, ψ is one-to-one, and onto. Therefore, ψ is an isomorphism. ■

5. Let G be a group of order $1225 = 5^2 \cdot 7^2$. Let $P \in \mathrm{Syl}_5(G)$ and $Q \in \mathrm{Syl}_7(G)$. Show the following. (20 pts)

(a) P is a normal subgroup of G.

Soln. Since $|G : N(P)| = |\mathrm{Syl}_5(G)| \equiv 1 \pmod 5$ and $|G : N(P)|$ is a divisor of 7^2, the only possibility is 1. Hence $\mathrm{Syl}_5(G) = \{P\}$. Since $xPx^{-1} \in \mathrm{Syl}_5(G)$, $xPx^{-1} = P$ for all $x \in G$ and $P \lhd G$. ■

(b) P is Abelian.

Soln. Suppose P is a group of order p^2 for a prime p. Since P is a nontrivial p-group, $|Z(P)| = p$ or p^2. If $|Z(P)| = p$, $P/Z(P)$ is a prime order and hence cyclic. Then P is Abelian and $P = Z(P)$, a contradiction. Thus $P = Z(P)$ and P is Abelian. Since $|P| = 5^2$, P is of prime square order, and hence P is Abelian. ■

(c) $G = P \times Q$.

Soln. Since $|G : N(Q)| = |\mathrm{Syl}_7(G)| \equiv 1 \pmod 7$ and $|G : N(Q)|$ is a divisor of 5^2, the only possibility is 1. Hence $\mathrm{Syl}_5(G) = \{Q\}$. Since $xQx^{-1} \in \mathrm{Syl}_5(G)$, $xQx^{-1} = Q$ for all $x \in G$ and $Q \lhd G$. Since each element of $P \cap Q$ has order dividing $|P| = 5^2$ and $|Q| = 7^2$ by Lagrange's Theorem, $P \cap Q = \{e\}$. Since $PQ = QP$, PQ is a subgroup of G by 1(d) containing P and Q as subgroups, $|PQ|$ is divisible by $5^2 7^2$ and $PQ = G$. Therefore, $G = P \times Q$, the internal direct product of P and Q. ■

(d) G has an element of order 35.

Soln. Since both P and Q are Abelian by (b), P contains an element x of order 5 and Q contains an element y of order 7. Since $G \approx P \oplus Q$, G contains an element of order 35 corresponding to xy. ■