Algebra I: Final 2017

June 21, 2017

ID#: **Name:**

Quote the following when necessary.

A. Subgroup H of a group G:

$$H \leq G \Leftrightarrow \emptyset \neq H \subseteq G,\ xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

B. Order of an Element: Let g be an element of a group G. Then $\langle g \rangle = \{g^n \mid n \in \boldsymbol{Z}\}$ is a subgroup of G. If there is a positive integer m such that $g^m = e$, where e is the identity element of G, $|g| = \min\{m \mid g^m = e,\ m \in \boldsymbol{N}\}$ and $|g| = |\langle g \rangle|$. Moreover, for any integer n, $|g|$ divides n if and only if $g^n = e$.

C. Lagrange's Theorem: If H is a subgroup of a finite group G, then $|G| = |G : H||H|$.

D. Normal Subgroup: A subgroup H of a group G is normal if $gHg^{-1} = H$ for all $g \in G$. If H is a normal subgroup of G, then G/H becomes a group with respect to the binary operation $(gH)(g'H) = gg'H$.

E. Direct Product: If $\gcd(m, n) = 1$, then $\boldsymbol{Z}_{mn} \approx \boldsymbol{Z}_m \oplus \boldsymbol{Z}_n$ and $U(mn) \approx U(m) \oplus U(n)$.

F. Isomorphism Theorem: If $\alpha : G \to \overline{G}$ is a group homomorphism, $\mathrm{Ker}(\alpha) = \{x \in G \mid \alpha(x) = e_{\overline{G}}\}$, where $e_{\overline{G}}$ is the identity element of $\overline{G}$. Then $G/\mathrm{Ker}(\alpha) \approx \alpha(G)$.

G. Sylow's Theorem: For a finite group G and a prime p, let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G. Then $\mathrm{Syl}_p(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. Then $|\mathrm{Syl}_p(G)| = |G : N(P)| \equiv 1 \pmod p$, where $N(P) = \{x \in G \mid xPx^{-1} = P\}$.

Other Theorems: List other theorems you applied in your solutions.

Please write your message: Comments on group theory. Suggestions for improvements of this course. Write on the back of this sheet is also welcome.

1. Let H and K be subgroups of a group G. Let $a, b \in G$. Show the following. (20 pts)

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.

   (b) $HH = H^{-1} = H$.

   (c) If $HK \leq G$, then $HK = KH$.

   (d) If $HK = KH$, then $HK \leq G$.

2. Let G be a finite group, $\phi : G \to H$ an onto group homomorphism, e_G is the identity element of G and e_H the identity element of H. Show the following. (20 pts)

   (a) $\phi(e_G) = e_H$ and for $a \in G$, $\phi(a^{-1}) = \phi(a)^{-1}$.

   (b) If K is a normal subgroup of H, then $\phi^{-1}(K) = \{x \in G \mid \phi(x) \in K\}$ is a normal subgroup of G.

   (c) $|\phi(x)| \mid |x|$ for all $x \in G$.

   (d) If H has an element of order n, then G has an element of order n.

3. Answer the following questions on Abelian groups of order $16 = 2^4$. (20 pts)

   (a) Using the Fundamental Theorem of Finite Abelian Groups, list all non-isomorphic Abelian groups of order 16.

   (b) Express $U(32)$ as an external direct product of cyclic groups. Show work.

   (c) Express $U(32)$ as an internal direct product of cyclic subgroups.

   (d) Is $U(40)$ isomorphic to $U(32)$? Show work.

4. Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \boldsymbol{R}, ad - bc \neq 0 \right\}$, and $B = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, d \in \boldsymbol{R}, ad \neq 0 \right\}$.
Show the following. (20 pts)

(a) B is not a normal subgroup of G.

(b) $\phi : B \to \boldsymbol{R}^* \oplus \boldsymbol{R}^* \left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto (a, d) \right)$ is a group homomorphism.

(c) $U = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \middle| b \in \boldsymbol{R} \right\}$ is a normal subgroup of B.

(d) $B/U \approx R^* \oplus R^*$.

5. Let G be a group of order $175 = 5^2 \cdot 7$. Let $P \in \mathrm{Syl}_5(G)$ and $Q \in \mathrm{Syl}_7(G)$. Show the following. (20 pts)

   (a) P is a normal subgroup of G.

   (b) P is Abelian.

   (c) Q is a normal subgroup of G.

   (d) G is Abelian.

Algebra I: Solutions to Final 2017

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1. Let H be a subgroup of a group G. Let $a, b \in G$. Show the following. (20 pts)

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.
   Soln. Since $H \leq G$, $H \neq \emptyset$. Let $a \in H$. Then $a^{-1} \in H$ and $e = aa^{-1} \in H$.
   Suppose $aH = bH$. Since $e \in H$, $aH = bH$ implies that $b = be \in bH = aH$. Hence there exists $h \in H$ such that $b = ah$. Therefore by multiplying a^{-1} to both hand sides from the left, $a^{-1}b = h \in H$.
   Conversely let $a^{-1}b = h \in H$. Then $b = ah$ and

   $$bH = ahH \subseteq aH = aeH = ahh^{-1}H = aa^{-1}bh^{-1}H \subseteq bH.$$

   Therefore $aH = bH$. ■

   (b) $HH = H^{-1} = H$.
   Soln. Since H is a subgroup of G, the identity element $e \in H$, for all $x, y \in H$, $xy \in H$ and $x^{-1} \in H$. Moreover, $x = (x^{-1})^{-1} \in H^{-1}$. Thus, $H \subseteq H^{-1}$ and

   $$H = eH \subseteq HH \subseteq H \subseteq H^{-1} \subseteq H.$$

   Therefore, $HH = H^{-1} = H$. ■

   (c) If $HK \leq G$, then $HK = KH$.
   Soln. Since both H and K are subgroups of G, by (b)

   $$HK = (HK)^{-1} = K^{-1}H^{-1} = KH.$$

   ■

   (d) If $HK = KH$, then $HK \leq G$.
   Soln. We use the two step subgroup test.

   $$HKHK = HHKK = HK, \quad (HK)^{-1} = K^{-1}H^{-1} = KH = HK.$$

   Hence, $HK \leq G$. ■

2. Let G be a finite group, $\phi : G \to H$ an onto group homomorphism, e_G is the identity element of G and e_H the identity element of H. Show the following. (20 pts)

   (a) $\phi(e_G) = e_H$ and for $a \in G$, $\phi(a^{-1}) = \phi(a)^{-1}$.
   Soln. $\phi(e_G) = \phi(e_G)^{-1}\phi(e_G)\phi(e_G) = \phi(e_G)^{-1}\phi(e_Ge_G) = \phi(e_G)^{-1}\phi(e_G) = e_H$. $\phi(a^{-1}) = \phi(a^{-1})\phi(a)\phi(a)^{-1} = \phi(a^{-1}a)\phi(a)^{-1} = \phi(e_G)\phi(a)^{-1} = e_H\phi(a)^{-1} = \phi(a)^{-1}$.
   ■

(b) If K is a normal subgroup of H, then $\phi^{-1}(K) = \{x \in G \mid \phi(x) \in K\}$ is a normal subgroup of G.

Soln. Let $a, b \in \phi^{-1}(K)$. Then $\phi(ab) = \phi(a)\phi(b) \in KK = K$. Hence $ab \in \phi^{-1}(K)$. By (a) $\phi(a^{-1}) = \phi(a)^{-1} \in K^{-1} = K$. Hence $a^{-1} \in \phi^{-1}(K)$. Thus $\phi^{-1}(K)$ is a subgroup of G. Let $g \in G$, then $\phi(gag^{-1}) = \phi(g)\phi(a)\phi(g^{-1}) \in \phi(g)K\phi(g)^{-1} = K$. Hence $gKg^{-1} \subseteq \phi^{-1}(K)$ for all $g \in G$. Since this holds for $g^{-1} \in G$, $g^{-1}\phi^{-1}(K)g \subseteq \phi^{-1}(K)$, which implies $\phi^{-1}(K) \subseteq g\phi^{-1}(K)g^{-1}$. Thus $g\phi^{-1}(K)g^{-1} = \phi^{-1}(K)$ for all $g \in G$ and $\phi^{-1}(K)$ is a normal subgroup of G. ■

(c) $|\phi(x)| \mid |x|$ for all $x \in G$.

Soln. Let $n = |x|$. Since $|G|$ is finite, n is finite. Then $x^n = e$ and $\phi(x)^n = \phi(x^n) = \phi(e_G) = e_H$. Thus by B, $|\phi(x)| \mid |x|$. ■

(d) If H has an element of order n, then G has an element of order n.

Soln. Let y be an element of order n in H, Since ϕ is onto, there is $x \in G$ such that $\phi(x) = y$. By (c) above, $n \mid |x|$. Let $m = |x|$. Since G is finite, m is finite. Let $h = m/n$. Then $|x^h| = n$ as $(x^h)^n = x^{hn} = x^m = e_G$ and $e, x, x^2, \ldots, x^{n-1}$ are all distinct. ■

3. Answer the following questions on Abelian groups of order $16 = 2^4$. (20 pts)

(a) Using the Fundamental Theorem of Finite Abelian Groups, list all non-isomorphic Abelian groups of order 16.

Soln.

$$\boldsymbol{Z}_{16}, \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_8, \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_4, \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4, \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2.$$

(b) Express $U(32)$ as an external direct product of cyclic groups. Show work.

Soln. Since $|U(32)| = 16$, the order of an element is 1, 2, 4, 8 or 16.

$$3^2 = 9, 3^4 = 17, 3^8 = 1, 31^2 = 1.$$

Hence $U(32) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_8$. ■

(c) Express $U(32)$ as an internal direct product of cyclic subgroups.

Soln.

$$U(32) = \langle 3 \rangle \times \langle 31 \rangle.$$

(d) Is $U(40)$ isomorphic to $U(32)$? Show work.

Soln.

$$U(40) \approx U(5) \oplus U(8) \approx \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2.$$

Hence $U(40) \not\approx U(32)$. ■

4. Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \boldsymbol{R}, ad - bc \neq 0 \right\}$, and $B = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, d \in \boldsymbol{R}, ad \neq 0 \right\}$. Show the following. (20 pts)

(a) B is not a normal subgroup of G.

Soln.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \not\in B.$$

(b) $\phi : B \to \boldsymbol{R}^* \oplus \boldsymbol{R}^* \left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto (a, d)\right)$ is a group homomorphism.

Soln.

$$\begin{aligned} \phi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} aa' & ab' + bd' \\ 0 & dd' \end{bmatrix}\right) = (aa', dd') = (a, b)(a', d') \\ &= \phi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right)\phi\left(\begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}\right). \end{aligned}$$

(c) $U = \left\{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \middle| b \in \boldsymbol{R}\right\}$ is a normal subgroup of B.

Soln. $U = \mathrm{Ker}\phi \lhd G$. Note that $\langle(1, 1)\rangle \lhd \boldsymbol{R}^* \oplus \boldsymbol{R}^*$ and the normality is a consequence of 2 (b). ∎

(d) $B/U \approx R^* \oplus R^*$.

Soln. Since ϕ is onto and the kernel is U, the assertion follows from the first isomorphism theorem. ∎

5. Let G be a group of order $175 = 5^2 \cdot 7$. Let $P \in \mathrm{Syl}_5(G)$ and $Q \in \mathrm{Syl}_7(G)$. Show the following. (20 pts)

(a) P is a normal subgroup of G.

Soln. Since $|G : N(P)| = |\mathrm{Syl}_5(G)| \equiv 1 \pmod 5$ and $|G : N(P)|$ is a divisor of 7, the only possibility is 1. Hence $\mathrm{Syl}_5(G) = \{P\}$. Hence $xPx^{-1} = P$ for all $x \in G$ and $P \lhd G$. ∎

(b) P is Abelian.

Soln. Since 5 is prime and G is of prime square order, G is Abelian. ∎

(c) Q is a normal subgroup of G.

Soln. Since $|G : N(Q)| = |\mathrm{Syl}_7(G)| \equiv 1 \pmod 7$ and $|G : N(Q)|$ is a divisor of 5^2, i.e., $1, 5$ or 5^2, the only possibility is 1. Hence $\mathrm{Syl}_7(G) = \{Q\}$. As in (a), $Q \lhd G$. ∎

(d) G is Abelian.

Soln. Since $P \lhd G$, $PQ = QP$ and $PQ \leq G$. We have $P \lhd PQ$ and $Q \lhd PQ$ by (a) and (c). Since $P \cap Q$ is a subgroup of P and Q of order 25 and 7 respectively, $P \cap Q = \{e\}$ by Lagrange's Theorem. Hence $PQ = P \times Q$, In particular, $|PQ| = 175$ and $G = PQ = P \times Q$. Since both P and Q are Abelian, G is Abelian. ∎