Algebra I: Final 2005 Solutions

June 23, 2005

1. Let H be a nonempty subset of a group G satisfying the following.

$$x^{-1}y \in H \text{ for all } x, y \in H.$$

   (a) Show that $1 \in H$, $x^{-1} \in H$ and $xy \in H$ for all $x, y \in H$. (This proves that $H \leq G$.)
   Solution. Since $H \neq \emptyset$, we can take an element $x \in H$. By the condition, $1 = x^{-1}x \in H$. Let x be an arbitrary element of H. Since $x, 1 \in H$, $x^{-1} = x^{-1}1 \in H$. For $x, y \in H$, since $x^{-1} \in H$, $xy = (x^{-1})^{-1}y \in H$. Therefore $1 \in H$, $x^{-1} \in H$ and $xy \in H$ for all $x, y \in H$. ■

   (b) Suppose $xhx^{-1} \in H$ for all $x \in G$ and $h \in H$. Show that $xHx^{-1} = H$ for all $x \in G$.
   Solution. By assumption $xHx^{-1} \subseteq H$ for every $x \in G$. Since $x^{-1} \in G$, $x^{-1}Hx = x^{-1}H(x^{-1})^{-1} \subseteq H$. Thus $xHx^{-1} = H$ for all $x \in G$. ■

2. Let H be a subgroup of a group G and N a normal subgroup of G. Let α be a mapping defined by

$$\alpha : H \to G/N \ (h \mapsto hN).$$

   (a) Show that α is a group homomorphism and that $\mathrm{Ker}(\alpha) = H \cap N$.
   Solution. Since N is a normal subgroup of G, G/N is a group with respect to the multiplication defined by $(h_1N)(h_2N) = h_1h_2N$. (Note that $(h_1N)(h_2N) = h_1(Nh_2)H = h_1h_2NN = h_1h_2N$.) Hence for $h_1, h_2 \in H$,

$$\alpha(h_1h_2) = h_1h_2N = (h_1N)(h_2N) = \alpha(h_1)\alpha(h_2).$$

   Thus α is a group homomorphism. Hence its kernel is normal and that

$$\begin{aligned} \mathrm{Ker}(\alpha) &= \{h \in H \mid \alpha(h) = 1_{G/N}\} = \{h \in H \mid hN = N\} \\ &= \{h \in H \mid h \in N\} = H \cap N. \end{aligned}$$

   Note that since $N \leq G$, $hN = N \Leftrightarrow h \in N$. ■

   (b) Show that $H/H \cap N \simeq HN/N$.
   Solution. <u>*Claim.*</u> $\alpha(H) = HN/N$.
   Proof. Since N is a normal subgroup of G, $HN = NH$. Thus $N \lhd HN \leq G$.

$$\alpha(H) = \{hN \mid h \in H\} = \{hnN \mid h \in H, n \in N\} = HN/N.$$

   Now by the First Isomorphism Theorem (4.3.4) and 2 (a),

$$H/H \cap N = H/\mathrm{Ker}(\alpha) \simeq \mathrm{Im}(\alpha) = HN/N$$

   as desired. ■

3. Let $(\boldsymbol{Z}_{14}^*, \cdot)$ be a multiplicative group; here $\boldsymbol{Z}_{14}^*$ is the set of invertible congruence classes $[a]$ modulo 14, i.e., such that $\gcd\{a, 14\} = 1$.

   (a) Show that $\boldsymbol{Z}_{14}^*$ is a cyclic group.
   Solution. By definition,

$$\boldsymbol{Z}_{14}^* = \{[1], [3], [5], [9], [11], [13]\},$$

   and $[3]^2 = [9]$, $[3]^3 = [13]$, $[3]^4 = [11]$, $[3]^5 = [5]$ and $[3]^6 = [1] = [3]^0$. Hence $\boldsymbol{Z}_{14}^* = \{[3]^n \mid n \in \boldsymbol{Z}\} = \langle [3] \rangle$ and $\boldsymbol{Z}_{14}^*$ is a cyclic group. ■

(b) Show that $[a]^6 = [1]$ for all $[a] \in \boldsymbol{Z}_{14}^*$. Show also that $a^6 \equiv 1 \pmod{14}$ for all integers a such that $\gcd\{a, 14\} = 1$.

Solution. Since $\boldsymbol{Z}_{14}^*$ is a cyclic group of order 6, $[a] = [3]^i$ for some $i \in \boldsymbol{Z}$ by (a). Since $[3]^6 = [1]$, $[a]^6 = ([3]^i)^6 = ([3]^6)^i = [1]$. Moreover if $\gcd\{a, 14\} = 1$, $[a] \in \boldsymbol{Z}_{14}^*$, and $[a]^6 = [a^6] = [1]$. Therefore $a^6 \equiv 1 \pmod{14}$. ■

(c) Find a normal subgroup N in $\boldsymbol{Z}_{14}^*$ such that $\boldsymbol{Z}_{14}^*/N$ is a cyclic group of order 2.

Solution. Since $2 = |\boldsymbol{Z}_{14}^*/N| = |\boldsymbol{Z}_{14}^*|/|N| = 6/|N|$, $|N| = 3$. Hence by (4,1,6),

$$N = \langle [3]^{6/3} \rangle = \langle [3]^2 \rangle = \langle [9] \rangle = \{[1], [9], [11]\}.$$

Clearly $\boldsymbol{Z}_{14}^* = N \cup [3]N$ as $[3]N = \{[3], [13], [5]\}$, and $\boldsymbol{Z}_{14}^*/N$ is a cyclic group of order 2. ■

4. Let S_8 be the symmetric group of degree 8.

(a) Let $\text{sign} : S_8 \to \{\pm 1\}$ $(\pi \mapsto \text{sign}(\pi))$ be the sign function on S_8. Show that Ker(sign) is a normal subgroup of S_8 of index 2.

Solution. For $\sigma, \tau \in S_8$, $\text{sign}(\sigma\tau) = \text{sign}(\sigma)\text{sign}(\tau)$. Hence sign is a homomorphism from S_8 to a multiplicative group $\{\pm 1\}$. Hence $\text{Ker(sign)} \lhd S_8$. Since $\text{sign}((12)) = -1$, $\text{Im(sign)} = \{\pm 1\}$. Thus

$$2 = |\{\pm 1\}| = |S_8/\text{Ker(sign)}| = |S_8|/|\text{Ker(sign)}| = |S_8 : \text{Ker(sign)}|.$$

Thus the index of Ker(sign) in S_8 is 2. ■

(b) Let $\sigma = (1, 2, 3, 4)(5, 6, 7)(8) \in S_8$. Find $|\sigma|$ and $\text{sign}(\sigma)$, the order and the value of the sign function.

Solution. $|\sigma| = \min\{n \in \boldsymbol{Z} \mid (n > 0) \wedge (\sigma^n = id)\} = 12$. Since $\sigma = (1, 2, 3, 4)(5, 6, 7)(8) = (1, 4)(1, 3)(1, 2)(5, 7)(5, 6)$, $\text{sign}(\sigma) = (-1)^5 = -1$. ■

(c) What is the largest possible order of an element of S_8? How many elements of S_8 have this order?

Solution. The largest order is

$$\max\{lcm\{n_1, n_2, \ldots, n_r\} \mid n_1 + n_2 + \cdots + n_r = 8,\ n_1, n_2, \ldots, n_r \geq 1\}.$$

$(n_1, \ldots, n_r)$	lcm	$(n_1, \ldots, n_r)$	lcm	$(n_1, \ldots, n_r)$	lcm
(8)	8	$(4,4)$	4	$(3,2,2,1)$	6
$(7,1)$	7	$(4,3,1)$	12	$(3,2,1,1,1)$	6
$(6,2)$	12	$(4,2,2)$	4	$(3,1,1,1,1,1)$	3
$(6,1,1,1)$	6	$(4,2,1,1)$	4	$(2,2,2,2)$	2
$(5,3)$	15	$(4,1,1,1,1)$	4	$(2,2,2,1,1)$	2
$(5,2,1)$	10	$(3,3,2)$	6	$(2,2,1,1,1,1)$	2
				$(1,1,1,1,1,1,1,1)$	1

Thus the largest order is 15, and the number of elements having this order is

$$\binom{8}{5} \cdot (5-1)! \cdot (3-1)! = 8 \cdot 7 \cdot 6 \cdot 4 \cdot 2 = 2688.$$

Hear $\binom{8}{5}$ reads 8 choose 5 and it is same as ${}_8C_5$. ■.

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Please write your message: (1) Comments on group theory. Suggestions for improvements of this course.

(2) Comments on the educaion at ICU. Suggestions for improvements.